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A CANAL SURFACE CONTAINING FOUR STRAIGHT LINES

Abstract: *A canal surface is the envelope of spheres with centers traversing a spatial curve called spine curve. The spheres contact the envelope along so-called characteristics, which are circles in general. If a canal surface contains two lines, then the spine curve is located on the bisector of these lines which in the case of skew lines is an orthogonal hyperbolic paraboloid. There are trivial cases of canal surfaces with infinitely many lines, the right cylinders, the right cones, and the one-sheeted hyperboloids of revolution. The only nontrivial case of a canal surface through four straight lines, that are not the limits of characteristics, is related to a Plücker conoid. The four given lines must be concyclic generators, i.e., they intersect each tangent plane of the conoid in four points lying on a circle. We are going to analyse and visualize this particular canal surface.*

Key words: *Canal surface, spine curve, Plücker's conoid, pedal curve, concyclic generators.*

1. INTRODUCTION

A *canal surface* or *channel surface* $\mathcal{E}$ is the envelope of a smooth one-parameter set of spheres with centers traversing a spatial curve called *spine curve* or *directrix*. The radii of the spheres can vary along the spine curve. Each enveloping sphere of the one-parametric set contacts $\mathcal{E}$ along a circle called *characteristic* (Figure 1). Hence, $\mathcal{E}$ is traced by circles with the additional property that the tangent planes along each circle form a cone or cylinder or revolution. In limiting cases the sphere can become a plane with a line as characteristic.

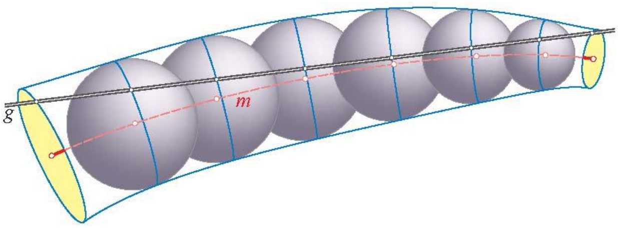

Figure 1 A canal surface as envelope of spheres which contact the line *g*. The centers of the characteristics form the displayed curve *m*.

Lemma 1. *If all spheres of a canal surface contact a line, then all points of contact belong to the surface.*

Proof. Given a canal surface $\mathcal{E}$, let P_1, P_2 be two sufficiently close points of the spine curve q. Then the characteristic of the sphere S_1 with center P_1 is the limit of the intersection with the sphere S_2 centered at P_2 when P_2 tends along q to P_1. The circle of intersection $S_1 \cap S_2$ lies in the plane of points with equal power with respect to (w.r.t., for short) the two spheres (note [1], p. 49).
Let T_1 and T_2 be the respective pedal points of the common tangent g w.r.t. P_1 and P_2. Then, S_1 passes through T_1 and S_2 through T_2. Moreover, the midpoint of the segment T_1T_2 has equal power w.r.t. S_1 and S_2. This means, that the midpoint is coplanar with the circle of intersection $S_1 \cap S_2$. Consequently, at the limit with $P_2 \to P_1$ and $T_2 \to T_1$ the point T_1 belongs to the characteristic of S_1. □

If all spheres of a canal surface contact two lines, then the spine curve must be located on the bisector of these lines, which in the case of skew lines is an orthogonal hyperbolic paraboloid and otherwise a pair of orthogonal planes (see Section 2.1). If all spheres contact three lines, then the spine curve belongs simultaneously to two bisectors.

There are trivial cases of canal surfaces which carry infinitely many lines: the right cylinders, the right cones, and the one-sheeted hyperboloids of revolution. In each case, the bisectors of any two lines meet at the axis.

The only nontrivial case of a canal surface where all spheres contact four straight lines[1] is related to a particular ruled surface of degree three, the Plücker conoid. As proved in [6], the four given lines must be concyclic generators of this surface. This means that they intersect each tangent plane of the conoid in four points lying on a circle and moreover on the ellipse, which is a component of the intersection between the conoid and the tangent plane (see Section 2.2).

The goal of this paper is to analyse and to visualize this particular canal surface (Section 3). At the begin we recall the necessary properties of the Plücker conoid.

2. PLÜCKER'S CONOID

A *Plücker conoid* $\mathcal{C}$ (also known under the name *cylindroid*) is a ruled surface of degree three with a finite double line and a director line at infinity (Figure 2). Using cylinder coordinates (r, φ, z), the conoid can be given by the equation

$$z = c \sin 2\varphi \qquad (1)$$

with a constant $c \in \mathbf{R} \setminus \{0\}$. All generators of $\mathcal{C}$ are parallel to the xy-plane. The z-axis is the double line d of $\mathcal{C}$ and an axis of symmetry. The conoid contains the x-axis and the

[1] A *parabolic Dupin ring cyclide* (see [3], Figure 10.18) contains four lines, but two of them are characteristics, one for each of the two possible generations of the cyclide as canal surfaces.

y-axis. These two lines c_1, c_2 are axes of symmetry of $\mathcal{C}$ and called *central generators*.

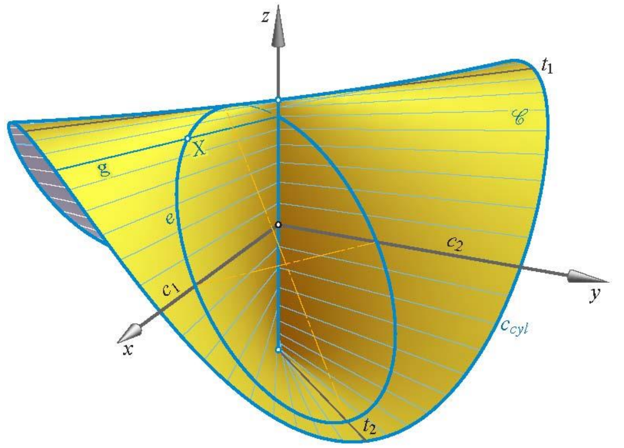

Figure 2 Plücker's conoid $\mathcal{C}$ with central generators c_1 and c_2, torsal generators t_1 and t_2, the generator g through X, and the ellipse e in the tangent plane τ_X to $\mathcal{C}$ at the point X.

The Plücker conoid $\mathcal{C}$ is traced by the x-axis under a motion which is composed from a rotation about the z-axis and a harmonic oscillation with double frequency along the z-axis [8].

The substitutions $x = r\cos\varphi$ and $y = r\sin\varphi$ in (1) yield the Cartesian equation

$$\mathcal{C}:\ (x^2 + y^2)\,z - 2cxy = 0, \qquad (2)$$

which reveals that reflections in the planes $x \pm y = 0$ map $\mathcal{C}$ onto itself. The origin is called the *center* of $\mathcal{C}$.

The right cylinder $x^2 + y^2 = R^2$ intersects the Plücker conoid $\mathcal{C}$ along a curve c_{cyl} of degree four[2] (see Figure 2), which in the cylinder's development appears as the Sine-curve with amplitude c and wavelength $R\pi$. The generators of $\mathcal{C}$ connect opposite points of c_{cyl}.[3] The conoid is bounded by the planes $z = \pm c$, which contact $\mathcal{C}$ along the *torsal generators* t_1 and t_2. Their distance $|2c|$ is called the *width* of $\mathcal{C}$.

For the sake of simplicity, we assume that the xy-plane and all generators of $\mathcal{C}$ are horizontal and the z-axis is vertical. In this sense, the *top view* stands for the image under vertical projection into the xy-plane; a prime will be used to indicate the top views of geometric objects.

The top view in Figure 3 (left) reveals that the intersection of the Plücker's conoid $\mathcal{C}$ with any right cylinder $\mathcal{Z}$ through the double line d gives a curve e which in the cylinder's development shows up as one period of a Sine curve. Therefore, e is an ellipse with principal vertices on the torsal generators.

There exists a two-parameter set of ellipses e on the conoid $\mathcal{C}$. They all have the same linear eccentricity c, as it equals the difference of the respective z-coordinates of a principal vertex and the center of e ([2], p. 208). The secondary vertices of e are located on the central generators c_1 and c_2. Ellipses $e \subset \mathcal{C}$ with the same minor semiaxis are congruent, and their planes have the same slope.

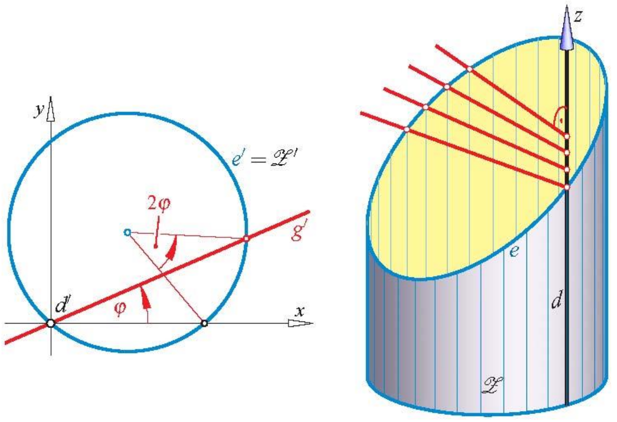

Figure 3 The intersections of the conoid $\mathcal{C}$ with right cylinders $\mathcal{Z}$ through the double line d are ellipses e with the same eccentricity. The generators of $\mathcal{C}$ meet e and intersect d orthogonally (left: top view, right: axonometric view)

Lemma 2. *Let g_1, g_2, g_3 be three non-coplanar lines with an orthogonal transversal d such that no two of the three lines are parallel. Then, there exists a unique Plücker conoid $\mathcal{C}$ passing through these lines.*

Proof. We choose a right cylinder $\mathcal{Z}$ which passes through d and does not contact any of the given lines. Then their remaining points of intersection with $\mathcal{Z}$ span a plane that intersects $\mathcal{Z}$ along an ellipse e thus defining $\mathcal{C}$ as shown in Figure 3. □

The remaining intersection between the cubic surface $\mathcal{C}$ and the plane of any ellipse $e \subset \mathcal{C}$ must be a line g passing through the common point of e and d (Figure 2). This generator g, which is horizontal and therefore parallel to the minor axis of e, shares with e another point X. This must be the point of contact between the conoid and the plane of e. In other words: The tangent plane τ_X to $\mathcal{C}$ at X intersects $\mathcal{C}$ beside the generator g along an ellipse e which appears in the top view as a circle $e' = \mathcal{Z}'$ through d'.

The top view gives insight into another important property of the ellipse $e \subset \tau_X \cap \mathcal{C}$ (Figure 4). For all points P in space with the top view $P' \in e'$ opposite to the top view d' of the double line, the *pedal curve* on $\mathcal{C}$, i.e., the locus of pedal points of P on the generators of $\mathcal{C}$, coincides with e. This holds since the right angles enclosed with generators of $\mathcal{C}$ appear in the top view again as right angles, provided that the spanned plane is not parallel to the double line d. It means conversely that for each point of e the surface normal to $\mathcal{C}$ meets the vertical line through P'. We summarize.

[2] The remaining part of the curve of intersection consists of the lines at infinity of the two complex conjugate planes $x \pm iy = 0$.

[3] See models #96 – #100 of the collection of mathematical models at the Institute of Discrete Mathematics and Geometry, Vienna University of Technology, https://www.geometrie.tuwien.ac.at/modelle/models_show.php?mode=2&n=100&id=0, accessed 2024-02-20. All models originate from Schilling's collection [5].

Lemma 3. *All pedal curves of Plücker's conoid $\mathcal{C}$ are planar. For points outside the double line the pedal curves are ellipses with the same eccentricity.*

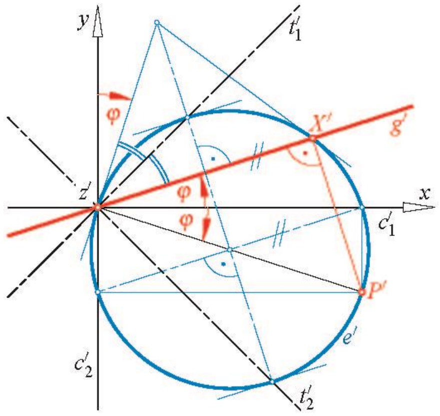

Figure 4 The point X is the pedal point of the generator g with respect to all points P with the top view P'; the ellipse $e \subset \mathcal{C}$ is the pedal curve of P.

2.1 Bisector of two skew lines

A classical result states that the *bisector* of two skew lines ℓ_1, ℓ_2, i.e., the set of points X being equidistant to ℓ_1 and ℓ_2, is an orthogonal (or equilateral) hyperbolic paraboloid (Figure 5). This is reported, e.g., in [4], p. 154, or in [3], p. 64.

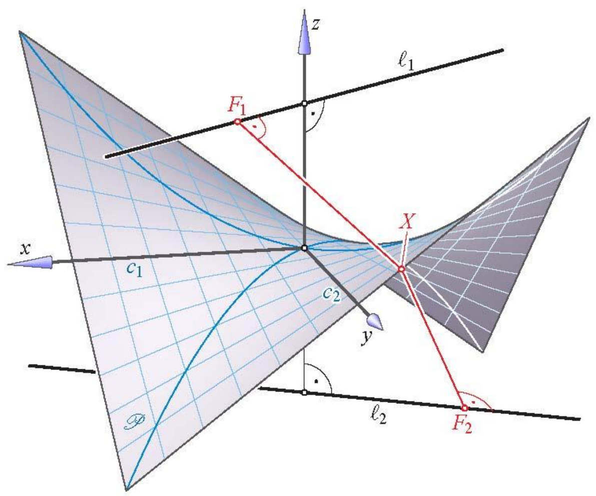

Figure 5 The bisector of two skew lines ℓ_1 and ℓ_2 is an orthogonal hyperbolic paraboloid $\mathcal{P}$ which contains the axes of symmetry c_1, c_2 of ℓ_1 and ℓ_2 as vertex generators.

Suppose that the two lines ℓ_1, ℓ_2 are given by $z = \pm d$ and $x \sin \varphi = \pm y \cos \varphi$. Then the distance of any space point $X = (x, y, z)$ to ℓ_i satisfies

$$[d(X\ell_i)]^2 = x^2 + y^2 + (z \mp d)^2 - (x \cos \varphi \pm y \sin \varphi)^2. \quad (3)$$

Consequently, the bisector $\mathcal{P}$ of the two lines is defined by the equation $[d(X\ell_1)]^2 - [d(X\ell_2)]^2 = 0$, i.e.,

$$\mathcal{P}\colon\ 2dz + xy \sin 2\varphi = 0. \quad (4)$$

Conversely, the question for all pairs (ℓ_1, ℓ_2) of lines for which a given orthogonal hyperbolic paraboloid $\mathcal{P}$ is the bisector, was already answered in [5], p. 54. Their geometric locus is the Plücker conoid

$$\mathcal{C}\colon\ z = c \sin 2\varphi \quad \text{for} \quad c = \frac{d}{\sin 2\varphi} \quad (5)$$

with the vertex generators of $\mathcal{P}$ as central generators and the axis of $\mathcal{P}$ as double line. For further details see [7].

2.2 Concyclic generators of Plücker's conoid

Given a Plücker conoid $\mathcal{C}$, let the ellipse $e \subset \mathcal{C}$ be the pedal curve of a point P. If a sphere S with the center P contacts some generators of $\mathcal{C}$, then their pedal points w.r.t. P must have equal distances to P. Since they are located in the plane of e, they belong to a circle $k \subset S$ with an axis through P (compare with Figure 6). The circle k can share at most four points with the ellipse e. Therefore, at most four generators of $\mathcal{C}$ can contact a sphere with the center P.

Definition 1. Four mutually different lines $g_1, \ldots, g_4$ are called *concyclic* if they belong to a Plücker conoid $\mathcal{C}$ and their points of intersection with any tangent plane τ_X to $\mathcal{C}$ are concyclic, i.e., located on a circle.

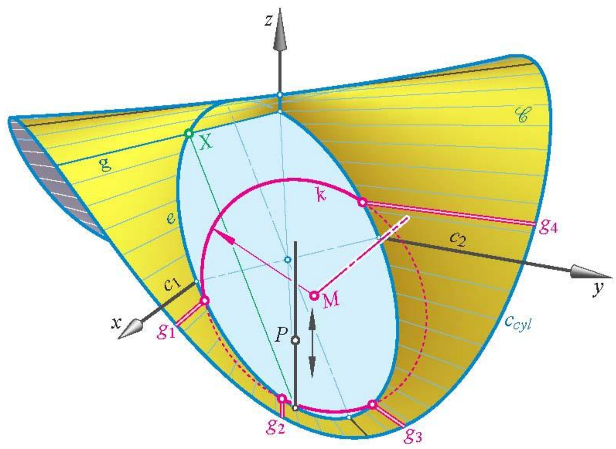

Figure 6 The four generators $g_1, \ldots, g_4$ of $\mathcal{C}$ are concyclic. There exist infinitely many spheres that contact these generators. The displayed point P with the pedal curve $e \subset \tau_X$ cannot be the center of one of these spheres since P is not located on the axis of the circle k, i.e., M does not belong to the vertical plane through P and the contact point X.

Lemma 4. *If the generators $g_1, \ldots, g_4 \subset \mathcal{C}$ are concyclic, then they intersect all tangent planes τ_X to $\mathcal{C}$ at four concyclic points, provided that in the particular case $g_i \subset \tau_X$ the point of contact X with $\mathcal{C}$ serves as the point of intersection $g_i \cap \tau_X$.*

Sketch of a *proof:* The top view reveals that generators of $\mathcal{C}$ intersects any two tangent planes of $\mathcal{C}$ at points which are corresponding under an affine transformation. Hence, the common points between the circle k and the ellipse e in the plane τ_X are sent to common points between two ellipses in any other tangent plane. It turns out that the pencil spanned by these two ellipses contains again a circle. This is since all pedal curves on $\mathcal{C}$ share the linear eccentricity. For details of the proof, the reader is referred either to [6], p. 60 or to [7].[4] □

Theorem 1. *If four lines $g_1, ..., g_4$ are concyclic, then there exist infinitely many spheres which contact these lines.*

Remark 1. According to [6], Satz 4, there are only two cases where four mutually skew lines $g_1, ..., g_4$ have a continuum of contacting spheres: The given lines are either concyclic or they belong to a hyperboloid of revolution. In the first case, the six bisecting hyperbolic paraboloids of the pairs (g_i, g_j) with $i, j \in \{1, ..., 4\}$, $i \neq j$, belong to a pencil. In the latter case, the paraboloids share the hyperboloid's axis.

Proof. By virtue of Lemma 2, the first three given lines g_1, g_2, g_3 with the common orthogonal transversal d define a Plücker conoid $\mathcal{C}$. The bisecting hyperbolic paraboloids of the pairs (g_1, g_2) and (g_1, g_3) share the vertical axis d and intersect each of the infinitely many horizontal planes along two concentric orthogonal hyperbolas with different asymptotes. These hyperbolas must meet at two real diametrical points which are centers of spheres tangent to the three lines. The three pedal points w.r.t. such a center P span a plane which contains the pedal curve e of P on $\mathcal{C}$. On the other hand, the contacting sphere $\mathcal{S}$ with center P intersects the plane of e along the circumcircle k of the three pedal points. The circle k and the ellipse e share a fourth point, and by virtue of Lemma 4 this point belongs to the fourth given generator $g_4 \subset \mathcal{C}$ which contacts the sphere $\mathcal{S}$, as well. □

Suppose that two out of four concyclic lines intersect each other on d. Then also the remaining two lines must intersect, since in this case the center of the circumcircle k is located on the principal axis of the ellipse $e = \tau_X \cap \mathcal{C}$. We call this an *intersecting case* in contrast to the *skew cases*. Another symmetric position of the concyclic lines arises when the circle k is centered on the secondary axis of e. This position is again independent of the choice of the plane τ_X.

3. THE ENVELOPE OF SPHERES THAT CONTACT FOUR CONCYCLIC LINES

By virtue of Lemma 1, a canal surface whose spheres contact simultaneously four lines, passes through these four lines. Hence, the only non-trivial case is the envelope $\mathcal{E}$ of the spheres as mentioned in Theorem 1.

Theorem 2. *Given four concyclic lines on the Plücker conoid $\mathcal{C}$, let $\mathcal{E}$ be the envelope of the contacting spheres.*
If the given lines are mutually skew, then the spine curve of $\mathcal{E}$ is a rational quartic q symmetric w.r.t. the double line d of $\mathcal{C}$ (Figure 7). The top view of q is an equilateral hyperbola with the top views of the torsal generators of $\mathcal{C}$ as asymptotes (Figure 8). The envelope $\mathcal{E}$ consists of two components which are symmetric w.r.t. d.

In the intersecting case, the spine curve of $\mathcal{E}$ splits into two parabolas in the vertical planes through the torsal generators. The parabolas are congruent and share the axis d.

Proof. If φ_i denotes for $i \in \{1, ..., 4\}$ the polar angle of g_i on the Plücker conoid $\mathcal{C}$ according to (1), then g_i has the z-coordinate $z_i := c \sin 2\varphi_i$. By (3) the distance of any space point $X = (x, y, z)$ to g_i satisfies

$$[d(Xg_i)]^2 = x^2 + y^2 + (z - z_i)^2 - (x \cos \varphi_i + y \sin \varphi_i)^2.$$

Hence, the bisecting paraboloid $\mathcal{P}_{ij}$ of the generators $g_i, g_j \subset \mathcal{C}$ obeys the equation $[d(Xg_i)]^2 - [d(Xg_j)]^2 = 0$, i.e.,

$$\mathcal{P}_{ij}: \ (\sin^2\varphi_i - \sin^2\varphi_j)(x^2 - y^2) - (z_i - z_j) + (xy/c + 2z) + (z_i^2 - z_j^2) = 0. \qquad (6)$$

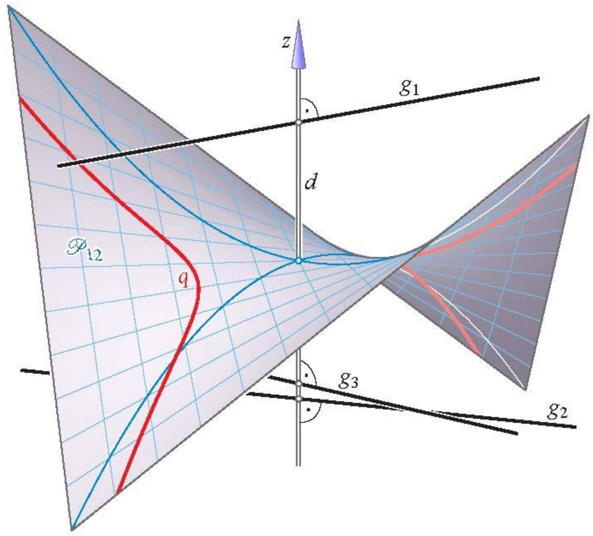

Figure 7 Spine curve q of the enveloping canal surface of all spheres that contact the skew lines g_1, g_2 (with bisector $\mathcal{P}_{12}$) and the line g_3.

All paraboloids with $z_i \neq z_j$ intersect the vertical planes through the torsal generators t_1, t_2 along congruent parabolas with the parameter $2c$. In the intersecting case with $z_1 = z_2$ and $z_3 = z_4$ the bisectors $\mathcal{P}_{12}$ and $\mathcal{P}_{34}$ split into the two vertical planes through t_1 and t_2. The other symmetric choice with $\varphi_2 = -\varphi_1$ and $\varphi_4 = -\varphi_3$ results in identical paraboloids $\mathcal{P}_{12} = \mathcal{P}_{34}$ satisfying the equation $xy + 2cz = 0$ in accordance with (5) and (4).
The bisectors $\mathcal{P}_{12}$ and $\mathcal{P}_{13}$ share a spatial curve q of degree four, and each point $P \in q$ is the center of a sphere $\mathcal{S}$ that contacts g_1, g_2 and g_3. As explained before, $\mathcal{S}$ must also contact the line g_4, which completes the concyclic quadruple. We obtain the equation of the top view q' of q as a linear combination of the equations of $\mathcal{P}_{12}$ and $\mathcal{P}_{13}$ after the elimination of z.

(a) In the skew case, the quartic q is irreducible. Its equation has the form

$$q': \ u\,(x^2 - y^2) = v \ \text{ with } \ u,v \in \mathbf{R} \setminus \{0\}, \qquad (7)$$

[4] As proved in [6], the four lines $g_1, ..., g_4$ are concyclic if and only if the (4 x 5)-matrix with rows $(1, z_i, z_i^2, \cos 2\varphi_i, \sin 2\varphi_i)$ for $i = 1, ..., 4$ has a rank $\leqq 3$.

where:

$$u := z_1 (\sin^2 \varphi_2 - \sin^2 \varphi_3) + z_2 (\sin^2 \varphi_3 - \sin^2 \varphi_1) + z_3 (\sin^2 \varphi_1 - \sin^2 \varphi_2),$$
$$v := (z_2 - z_1)(z_3 - z_2)(z_1 - z_3).$$

Consequently, q' is an equilateral hyperbola with the semi-axis $\sqrt{|v/u|}$. The asymptotes $x \pm y = 0$ are the top views of the torsal generators t_1, t_2 of $\mathcal{C}$ (Figure 8). In order to compute the z-coordinate of the points of q, we use the equation of $\mathcal{P}_{12}$ which is linear in z. Therefore, q is rational.

(b) In the intersecting case, we can assume $z_1 = z_2 \neq z_3$. Then $\mathcal{P}_{12}$ splits into the vertical planes through t_1 and t_2. Each of them intersects $\mathcal{P}_{13}$ along a parabola with the vertical axis d. □

Each point P of the spine curve is the center of a sphere S which has a real contact with the envelope $\mathcal{E}$ along the circumcircle k of the pedal points of g_1, ..., g_4 w.r.t. P. The circle k is located in a tangent plane τ_X of $\mathcal{C}$. The generator $g \subset \tau_X$ is horizontal (note Figure 6). The contact point X belongs together with P to a vertical plane which is parallel to the principal axis of the ellipse $e \subset \tau_X$. This plane passes also through the center M of k which is the pedal point of the plane τ_X w.r.t. P. Therefore, the axis $[P,M]$ of the characteristic circle k, which is tangent to the spine curve q at P, has a top view that coincides with $[P',X']$.

3.1 Generic case

As mentioned above, each point of the spine curve q is the center of a sphere that contacts the given lines g_i for $i = 1, ..., 4$ at real points. However, the top view reveals that conversely not each point T of g_i needs to be a contact point with a real sphere of the one-parameter set. The line that connects T with the center $P \in q$ of the contacting sphere must have a top view $[T',P']$ which meets the equilateral hyperbola q' and is perpendicular to g_i'. This holds for all points $T' \in g_i'$ only if g_i' has no real intersection with q' (like g_2' and g_3' in Figure 8). Otherwise, there remains a segment on g_i' symmetric w.r.t. d' which is not part of $\mathcal{E}$ (note g_1' and g_4' in Figure 8). The terminating points are uniplanar points of $\mathcal{E}$, and the points T for which $[T',P']$ meets the equilateral hyperbola q' twice are biplanar. We can summarize:

Theorem 3. *Given four mutually skew concyclic lines g_1, ..., g_4 on the Plücker conoid $\mathcal{C}$, let $\mathcal{E}$ be the envelope of the contacting spheres.*

The vertical planes through the torsal generators of $\mathcal{C}$ separate the space into two pairs of opposite sectors. Any given line $g_i \subset \mathcal{C}$ belongs completely to the surface $\mathcal{E}$ and all points of g_i are regular for each component of $\mathcal{E}$ if and only if g_i lies in the sector which is disjoint to the spine curve q of $\mathcal{E}$.

Otherwise a segment of g_i symmetric w.r.t. the double line $d \subset \mathcal{C}$ lies in the exterior of $\mathcal{E}$, and two halflines along g_i are curves of self-intersection of the components of $\mathcal{E}$.

The example displayed in Figure 11 reveals that there exist cases where all four concyclic lines g_1, ..., g_4 belong completely to both components.

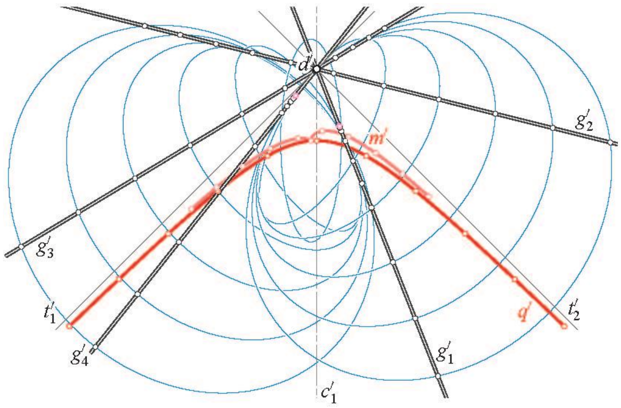

Figure 8 Top view of a sample of circles of one component of the canal surface $\mathcal{E}$ through the mutually skew concyclic lines g_1, ..., g_4 on a Plücker conoid with torsal generators t_1, t_2 and double line d. The hyperbola q' is the top view of the spine curve and m' that of the curve of circle centers.

We focus at first on the skew case and confine us to one connected component of the spine curve $q = \mathcal{P}_{12} \cap \mathcal{P}_{13}$, i.e., to one branch of the hyperbola q'. The complete canal surface contains a second component which is obtained by a halfturn about the double line d. The shape of the envelope $\mathcal{E}$ is hard to grasp due to possible singularities like biplanar and uniplanar points on the given lines (Theorem 3). Figure 9 shows a view of a singularity-free portion of this envelope $\mathcal{E}$. However, it can be shown that in the skew case the surface $\mathcal{E}$ always contains singularities in form of cuspidal edges.

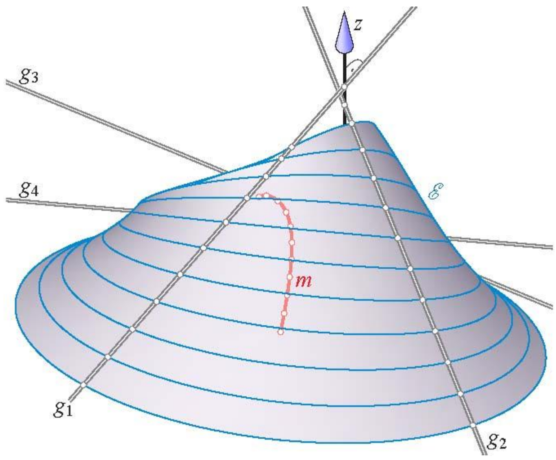

Figure 9 A portion of the canal surface $\mathcal{E}$ through the mutually skew concyclic lines g_1, ..., g_4 from Figure 8 along with the locus m of the characteristics' centers.

Theorem 5. *The center curve m of the circles k on the enveloping canal surface $\mathcal{E}$ is rational, too.*

Proof. Let $P = (\xi, \eta, \zeta)$ be any point in space. Firstly, we compute the corresponding pedal curve e on the Plücker conoid $\mathcal{C}$: The pedal points of the central generators c_1, c_2 w.r.t. P are the secondary vertices

$$C_1 = (\xi, 0, 0) \text{ and } C_2 = (0, \eta, 0)$$

of the ellipse e. This yields for the semiaxes a, b of the ellipse e

$$4b^2 = \xi^2 + \eta^2 \text{ and } a^2 = b^2 + c^2.$$

The principal vertices of e are the pedal points of the torsal generators t_1, t_2 w.r.t. P. Thus, we obtain

$$T_1 = ½\,(\xi + \eta,\ \xi + \eta,\ c) \quad \text{and} \quad T_2 = ½\,(\xi - \eta,\ \eta - \xi,\ -c).$$

The plane ε of e is spanned by the four vertices. The center M of the circle k is the pedal point of ε w.r.t. P. This yields after some computation

$$M = \frac{1}{4a^2b^2}\,(c^2\xi^3 + 2b^2c\eta\zeta + 4b^4\xi,\ c^2\eta^3 + 2b^2c\xi\zeta + 4b^4\eta,\ 2b^2c\,(2c\zeta + \xi\,\eta)).$$

If P runs along a rational curve like the quartic q, then m is rational, too (note the top view m' in Figure 8). This confirms the claim. □

3.2 Symmetric cases

We can distinguish between two symmetric cases where the centers M of the characteristic circles k are specified either on the principal axes (= slope lines) or on the secondary axes (= horizontal lines) of the respectively coplanar pedal curves, the ellipses $e \subset \mathcal{C}$. The cases of the first type are intersecting, the others are skew.

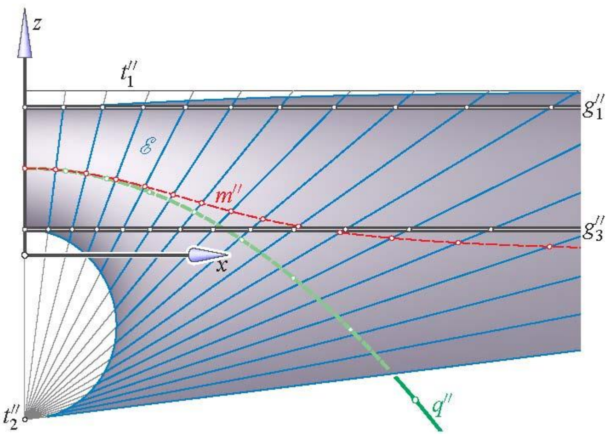

Figure 10 Orthogonal view of one quarter of the canal surface $\mathcal{E}$ in the intersecting case. The lines g_1 and g_2 have coinciding images as well as g_3 and g_4.

(a) In the intersecting case, the spine curve splits into two congruent parabolas (Theorem 2). One parabola opens to the positive z-axis, the other to the negative. We confine us to the latter and to one half of the parabola q in the vertical plane through the torsal line t_1. This plane is a plane of symmetry of the envelope $\mathcal{E}$, and Figure 10 shows $\mathcal{E}$ after the orthogonal projection into this plane. We speak of a front view and indicate this by two primes. Note that the images of the given lines g_1 and g_2 coincide as well as that of g_3 and g_4. The planes of the characteristic circles pass through the second torsal line t_2.

(b) Due to the symmetry in the skew case, the centers M of the circles k are always located on the secondary axes of the pedal curves e. Consequently, the point M traverses a planar curve in the xy-plane.

As shown in Figure 11, in this case the envelope has singularities in form of lines of regression like in all other skew cases. Apparently, at the specified example all points of the four lines are regular points of the displayed component of $\mathcal{E}$ (note Theorem 3). In this particular example, the two given lines g_2 and g_3 are rather close to the centerline c_1 of the Plücker conoid. Therefore, one component of the traces of $\mathcal{C}$ in the xy-plane looks almost aligned. However, it would result in a contradiction if $\mathcal{E}$ contains five lines.

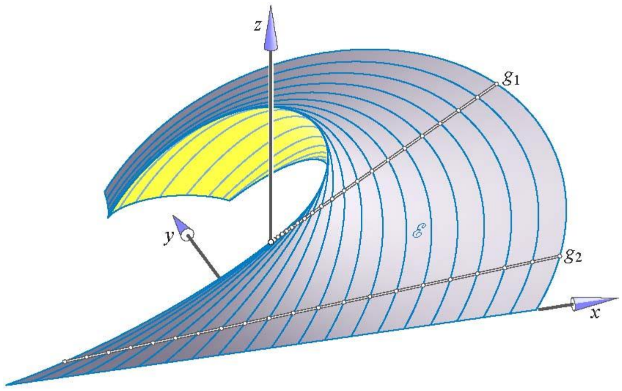

Figure 11 Upper half ($z > 0$) of one component of the envelope $\mathcal{E}$ in the skew symmetric case. The lower half with g_3, g_4 is symmetric w.r.t. the y-axis.

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