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THE HYPERBOLOID OF REVOLUTION OF ONE NAPPE – A RULED SURFACE GENERATING CONICS

Abstract: *The paper intends to underline the property of the hyperboloid of revolution of one nappe to generate on its surface, when it is cut by a plane, the three known conics (ellipse, hyperbola and parabola).Cutting this surface by a plane, we find the equation of the section to be the general equation of a conic. Depending on the parameters of this equation, we can establish the nature of the conic. The paper intends to emphasize the property of similarity between the ruled hyperboloidal surfaces of revolution and the conic ones, regarding their capacity to "house" conics on their surfaces; it also points out the existence of an alternative for the Dandelin's theorem in the case of the hyperboloid.*

Key words: *surface of revolution, ruled surface, hyperboloid, cone, asymptote, conics.*

INTRODUCTION

As it is well known, a surface of revolution is generated by rotating a curved line or a straight line generatrix round about a fixed axis.

If the generatrix is a straight line, the surface of revolution is called a ruled surface.

In fig. 1 there are three cases of ruled surfaces with the same axis Δ. If the generatrix D_1 intersects the axis Δ in the point S, the ruled surface will be a cone with the vertex in S.

If the generatrix D_2 is parallel with the axis Δ, the ruled surface will be a cylinder.

If the generatrix D_3 is not in the same plane with the axis Δ (D_3 and Δ are skew lines), the ruled surface will be a hyperboloid of revolution of one nappe.

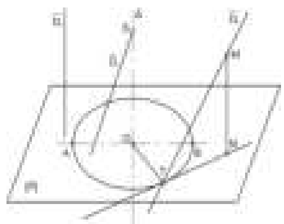

Fig. 1 The generation of three ruled surfaces

During the movement of revolution, each point of the generatrix describes a circle which plane is perpendicular to the axis of the surface. These circles are called "parallels" and the smallest of these parallels is known as the "gorge circle".

A plane passing through the axis of revolution intersects the surface by two curved or straight lines symmetric about the axis, called "meridians".

The representation of a surface of revolution implies two real size projections, one for the parallels, the other for the meridians.

1. REPRESENTING A HYPERBOLOID OF REVOLUTION

When the equation of the hyperbola is related to its axis, both axis Ox and Oy are axis of symmetry.

Because of this double symmetry of a hyperbola, there are two kinds of hyperboloids of revolution: the one of one nappe and the other of two nappes [1].

The one of two nappes is not a ruled surface, so it is not of our interest now.

The hyperboloid of revolution of one nappe is generated like in fig. 1.

When the line D_3 rotates about the axis Δ, OT is the common perpendicular of the lines Δ and D_3 (the shortest distance of the two lines).

Rotating D_3 about Δ, the point T describes the gorge circle of the surface.

Let's take M to be the current point of the surface.

If [P] is the plane of the gorge circle, N is the projection of M on the plane [P].

It is easy to prove (using the theorem of the three perpendiculars) that $NT \perp OT$ (because $MN \perp [P]$ and $MT \perp OT$).

That means that the projection of the generatrix D_3 on the plane [P] is tangent to the gorge circle.

The representation of the hyperboloid (fig. 2) implies two projections.

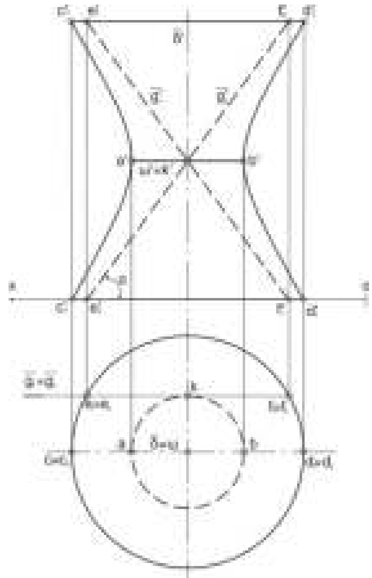

Fig. 2 The hyperboloid of revolution

The vertical projection consists of the two branches of the hyperbola (c'_1 a' c'_2) and (d'_1 b' d'_2), $a'b'$ being the projection of the gorge circle; e'_1 f'_1 and e'_2 f'_2 are the vertical projections of the two generatrices E_1F_1 and E_2F_2 which, during the revolution, became parallel with the vertical plane of projection (frontal lines).

The horizontal projection consists of the gorge circle of diameter ab and the great circles of diameters $c_1d_1 = c_2d_2$.

The horizontal projections of the frontal generatrices E_1F_1 and E_2F_2 are e_1f_1 and e_2f_2. Being frontal lines, their vertical projections e'_1 f'_1 and $e'_2f'_2$ give the real size of the angle α of a generatrix and the plane [P].

2. THE ANGLE OF THE GENERATRIX OF THE HYPERBOLOID AND THE PLANE OF THE GORGE. THE MERIDIAN

In fig. 3 the meridian of the hyperboloid is represented, Δ being the axis of the surface, G its generatrix and T the point of intersection between G and the gorge circle. Let′s project the current point M of the generatrix G on the plane [P] of the gorge circle in N.

If O is the center of the gorge circle, let′s note:

$ON = x$ and $MN = y$.

Let′s find now the locus of the point M, namely the equation of the meridian curve.

In the triangle MNT we have:

$y = NT\ \mathrm{tg}\alpha$

In the triangle OTN we have:

$NT^2 + r^2 = x^2$

Eliminating NT between the above relations we obtain:

$$\frac{y^2}{\mathrm{tg}^2\alpha} + r^2 = x^2 \qquad (1)$$

which gives:

$$\frac{x^2}{r^2} - \frac{y^2}{r^2 + \mathrm{tg}^2\alpha} = 1 \qquad (2)$$

This relation represents the equation of a hyperbola related to its center O and to its axis.

So, the meridian curve of a hyperboloid of rotation of one nappe is a hyperbola [1], [2], [3].

When the point M is in the vertical plane [V] of the representation (defined by the axis Δ and the diameter AB of the gorge circle), its projection N on the plane of the gorge will be situated on AB.

In this case, if the point M tends to infinity on the generatrix G in the direction of the asymptote A_1, the angle α of the generatrix G and the plane [P] of the gorge will be the same with the angle α of the asymptote A_1 and the diameter AB of the gorge circle (the axis Ox of the system of coordinates), measured in the vertical plane [V] of the representation.

It′s clear now that the angle α of any asymptote A with the axis Ox of the orthographic representation will be the same with the angle α of any generatrix G of the hyperboloid and the plane [P] of the gorge circle.

Rotating the asymptote A_1 about the axis Δ, this asymptote will generate a cone, named the cone asymptote of the hyperboloid. The vertex angle of this cone is: $\pi/2 - \alpha$. We shall see in the end of the paper the role of this cone in establishing the nature of the curve cut by a plane in the hyperboloid.

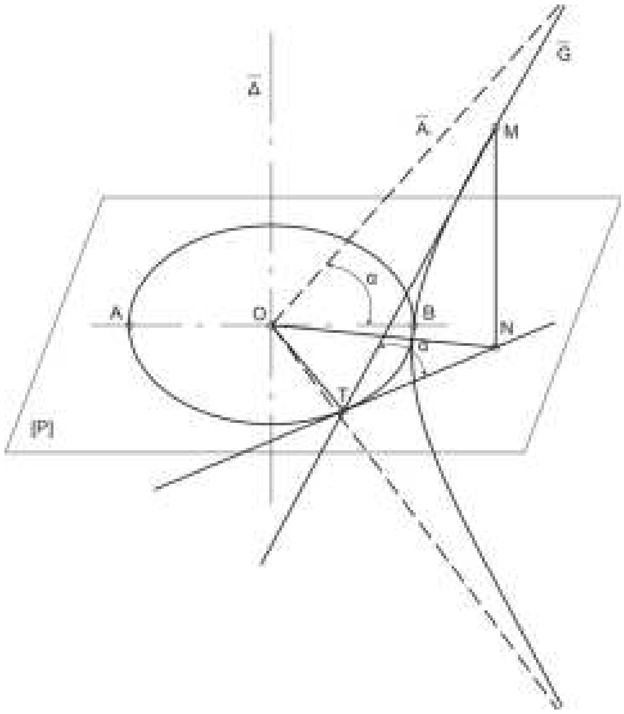

Fig. 3 The meridian of the hyperboloid

3. DETEREMINING THE EQUATION OF THE CURVE SECTIONED BY A PLANE IN THE HYPERBOLOID

Fig. 4 represents a hyperboloid of revolution of one nappe in two projections. The gorge circle has the diameter (ab, a′b′). The axis of the surface is (δ, δ′).

In the frontal plane f_H which contains the meridians of the hyperboloid, the asymptotes (a_1, a'_1) and (a_2, a'_2) intersect each other in the center of the hyperboloid (ω, ω′).

Let′s take a cutting plane [Q] (q_H, q'_V) perpendicular to [V].

We intend to find the equation of the curve of intersection of this plane [Q] and the hyperboloid.

As [Q] is a projecting plane with respect to the plane [V], the vertical projection of the cutting curve is a segment c′d′ laying on the vertical trace q'_v of the plane [Q].

The horizontal projection of this curve will indicate its nature.

Let′s take a current point M (m, m′) of the curve of section. We construct the tangent G (g, g′) to the gorge circle (ab, a′b′) passing through the point M (m, m′).

This tangent is in fact the generatrix of the hyperboloid MT, where T is the point of tangency of the generatrix and the gorge circle.

If we note:

$\Omega T = r$,

$\omega_x m_x = x$,

$m_x\ m = y$,

in the triangle: $\omega_x m_x m$ we have:

$$x^2 + y^2 = (\omega_x m)^2 \qquad (3)$$

in the triangle: ωtm we have:

$$(\omega m)^2 = r^2 + (mt)^2 \qquad (4)$$

in space, in the triangle m′tm we have:

$$\text{tg}\,\alpha = \frac{m'm_x}{mt} \qquad (5)$$

Also in space, in the triangle $m'q_xm$, we have:

$$\text{tg}\,\theta = \frac{m'm_x}{q_xm_x} = \frac{m'm_x}{x+d} \qquad (6)$$

The relations (3) and (4) give:

$$x^2 + y^2 = r^2 + (m_xm'\text{ctg}\,\alpha)^2 = r^2 + (x + d)^2\text{tg}^2\,\theta\text{ctg}^2\,\alpha$$

If we note: $k = \text{tg}\,\theta/\text{tg}\,\alpha$, the above relation becomes:

$$x^2 + y^2 = r^2 + k^2\,(x + d)^2 \qquad (7)$$

Changing the system of coordinates of the plane [H] with the one of the plane [Q], the relations between these two systems are:

$$x = x_1 \cos\theta \text{ and } y = y_1,$$

where x_1 and y_1 are the coordinates of the plane [Q].

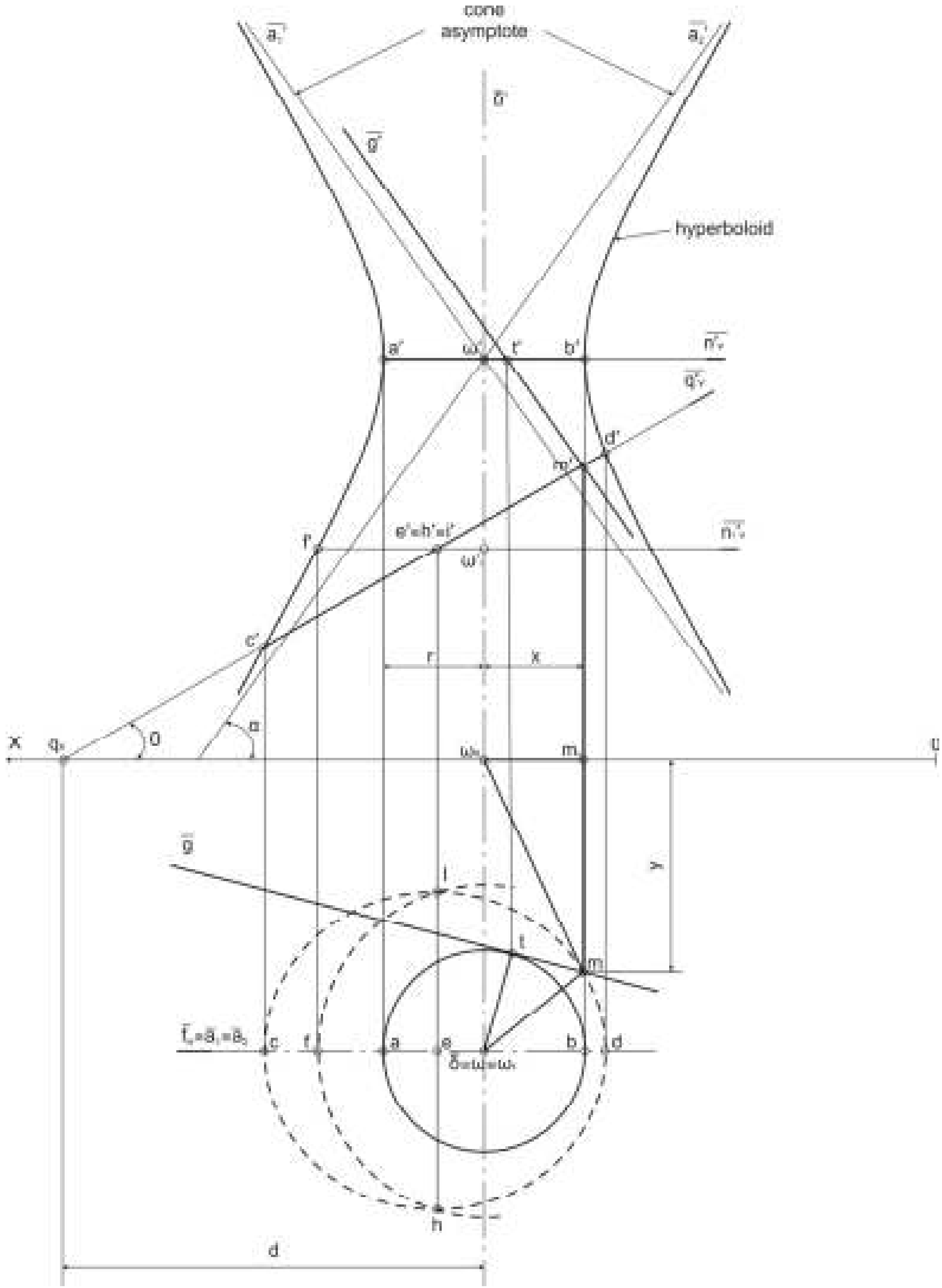

Fig. 4 The elliptic section in the hyperboloid of revolution of one nappe

These relations are like this because the plane [Q] is perpendicular to the plane [V] and the angle of these two planes is θ.

Using the new coordinates x_1 and y_1, the equation (7) of the curve becomes in the plane [Q]:

$$y_1^2 + [(1 - k^2) \cos^2 \theta] x_1^2 - (2k^2 d \cos \theta) x_1 - - (r^2 + k^2 d^2) = 0 \quad (8)$$

The equation (8) represents the general equation of a conic [1], [2], [3].

Depending on the sign of the quantity $(1 - k^2)$, the section is:

- ellipse, if $1 - k^2 > 0$, i.e. $\theta < \alpha$,
- hyperbola, if $1 - k^2 < 0$, i.e. $\theta > \alpha$, and
- parabola, if $1 - k^2 = 0$, i.e. $\theta = \alpha$.

If we recall now the Dandelin′s theorem regarding the right circular cone, we can establish a similar theorem regarding the hyperboloid of revolution of one nappe:

- if the plane [R], passing through the vertex of the cone asymptot of the hyperboloid and parallel to the cutting plane [Q], does not intersect this cone, the section made by [Q] in the hyperboloid will be an ellipse;
- if the plane [R] intersects the cone asymptot by two generatrices, the section made by [Q] in the hyperboloid will be a hyperbola;
- if the plane [R] is tangent to the cone asymptot, the section made by [Q] in the hyperboloid will be a parabola.

4. CONCLUSIONS

The paper uses the graphical methods of the descriptive geometry [4], [5], to establish the analytical equations of the curve cut by a plane in a hyperboloid of revolution of one nappe.

A parallelism between the conical and hyperboloidal surfaces regarding their property to generate conics is obvious.

Usualy, the ellipse, the hyperbola and the parabola are called conical curves, but we could name them hyperboloidal curves as well.

The equation (8) infered in cap. 4 proves that a random section cut buy a plane in a hyperboloid of revolution of one nappe represents a general conic (ellipse, hyperbola or parabola).

It is easy to deduce the Dandelin′s theorem for the hyperboloid of revolution of one nappe, using the cone asymptote of the hyperboloid.

Extending our studies in the direction indicated in our paper [6], we shall determine the specific elements of the three conics in future works, treating them as plane sections cut in a hyperboloid of revolution of one nappe.

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