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CYCLING CURVES AND THEIR APPLICATIONS

Abstract: *This paper proposes an analysis of the cyclic curves that can be considered as some of the most important regarding their applications in science, technique, design, architecture and art. These curves include the following: cycloid, epicycloid, hypocycloid, spherical cycloid and special cases thereof. In the first part of the paper the main curves of cycloids family are presented with their methods of generating and setting parametric equations. In the last part some of cycloid applications are highlighted in different areas of science, technology and art.*

Key words: *Circle, cycloid, epicycloid, hypocycloid, arc, rolling, parameter, equation*.

INTRODUCTION. BRIEF HISTORY

From the historical point of view, a cycloid is the representative curve of the family, by the interest aroused among mathematicians from the 16th and 17th centuries. It has helped to increase interest in the study of geometric properties of plane curves. Because of this fact and quarrels among mathematicians on the properties of cycloid, it was called „Helen of geometers" (in reference to the Helen of Troy).

Among mathematicians who have studied the cycloid we can mention: Nicolaus Cusanus (1401-1464); Marin Mersenne (1588-1648); Galileo Galilei (1564-1642) gave the current name of the curve in 1599; in 1634 Gilles Personne de Roberval (1602 – 1675) established that the area under a cycloid is three times the area of its generating circle; Christopher Wren (1632-1723) showed, in 1658 that the length of an arc of a cycloid is four times greater than the diameter of its generating circle; Blaise Pascal (1623-1662); Christiaan Huygens (1629-1495); Gottfried Wilhelm Leibnitz (1646-1716); Johann Bernoulli (1667-1748) and so on.

Some of these mathematicians went into posterity and because of their implication in the cycloid study. Thus, a marble statue of Augustin Pajou (1785), features Blaise Pascal studying the cycloid (work at the Louvre Museum, in 1960).

In a mezzotinto engraving by Johann Jakob Haid (1742), Johann Bernoulli is presented with a manuscript where you notice a cycloid [4].

2. CYCLOID

The cycloid is a curve described by a point *M* of a circle when it rolls over without slipping on a straight line, also called base.

In order to set parametric equations of cycloid you can choose a system reference with *Ox* axis which coincides with the fixed line (rolling line), with the rolling direction of the circle, and in its original position, the point *M* is the same as the origin *O* of a reference system (fig. 1).

The curve parameter (Γ_C) is the angle *t* called rollover angle, formed by the radius $r = MC$ and radius *CP*, where *M* is a point on the cycloid, *C* is the center of the generator circle, and *P* is the contact point of the circle to the base (fixed line).

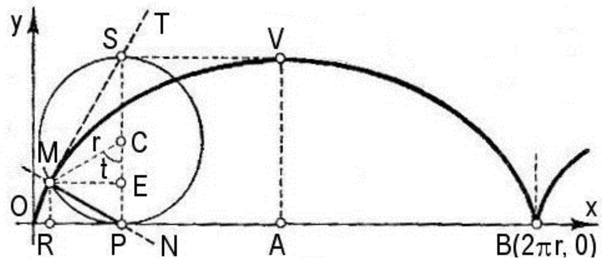


Fig. 1 Cycloid itself (common)

After calculatting the parametric equations of common cycloid are obtained:

$$(\Gamma_C):\begin{cases} x = r(t - \sin t) \\ y = r(1 - \cos t). \end{cases} \qquad (1)$$

A loop of cycloid (fig. 1) corresponds to $t \in [0, 2\pi]$ and in the remaining definition interval the other loops are also repeted [3].

The normal to the cycloid in the *M* point passing through the contact point *P* of the rolling circle to the basic line, while the tangent *MT* passes through a point *S*, which is a symmetric point to *P* point to the center of the circle that rolls over.

The area under a cycloid is $A = 3\pi r^2$, and the length of a cycloid arc is $L = 8r$, where *r* is the radius of the generating circle.

If point *M* does not belong to the circumference of the circle that rolls over, but it is a point inside the circle, then the curve is a shortened cycloid (fig. 2).

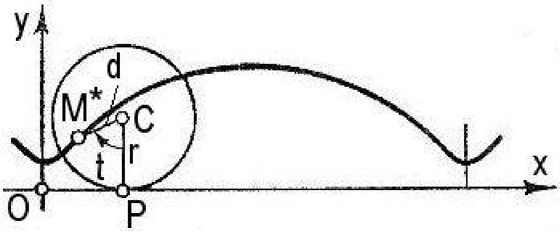


Fig. 2 Shortened cycloid

If point M is outside the circle, the curve obtained is an elongated cycloid (fig. 3).

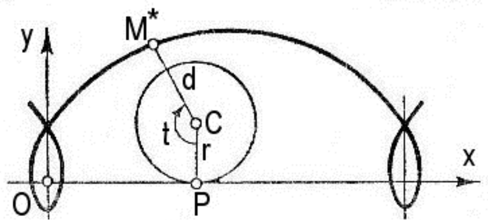


Fig. 3 Elongated cycloid

Noting with M^* the generating point in both cases and with d the distance from this point to the center of the generating circle we set parametric equations:

$$(\Gamma_T):\begin{cases} x = r\cdot t - d\cdot\sin t \\ y = r - d\cdot\cos t. \end{cases} \qquad (2)$$

In equation (2), for $d = r$ the common cycloid is obtained, for $d < r$ shortened cycloid and for $d > r$ elongated cycloid. These curves are called *trohoids*.

3. EPICYCLOID

An epicycloid is the curve described by a point M of a circle that rolls over without slipping on the outer side of a fixed circle.

For setting parametric equations, a fixed circle of radius R must have the center in the origin of reference system, and the axis Ox to go through initial position on the mobile circle that generates the epicycloid (fig. 4).

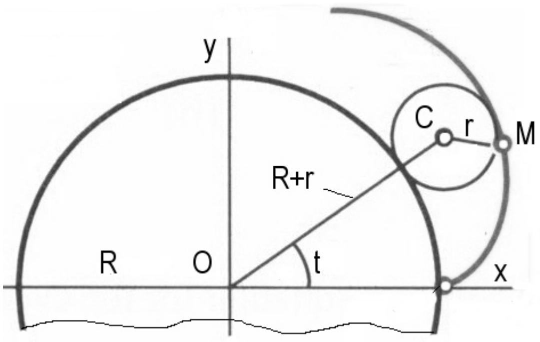


Fig. 4 Epicycloid

The parametric equations of the epicycloid are:

$$(\Gamma_{EC}):\begin{cases} x = (R+r)\cos t - r\cdot\cos\dfrac{R+r}{r}t \\ y = (R+r)\sin t - r\cdot\sin\dfrac{R+r}{r}t. \end{cases} \qquad (3)$$

Based on the location of the point M towards the centre of the generator circle we have: the common epicycloid ($d = r$), shortened epicycloid ($d < r$) and elongated epicycloid ($d > r$), where d is the distance from the point M to the center of the mobile circle. These curves are also called *epitrochoids.*

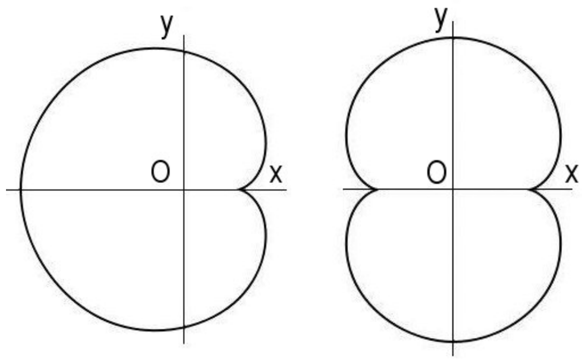
In relation to the fixed circle an epicycloid will have peaks, loops or minimals without double points. If the length $2\pi R$ is a multiple of the perimeter $2\pi r$, then the curve will have R/r arcs. If $k = R/r$ is an integer number the entire curve is closed and has k cusps, if k is a rational number the p/q curve has p cusps, and if k is irrational the curve is not closed but fills the space between the fixed circle and a circle of $R + 2r$ radius , concentric to the large circle.

If $k = R/r$ is an integer, the length of an epicycloid arc is $l = 8r(R + r)/R$, and the area between an epicycloid arc and fixes circle is $A = \pi r^2(3R + 2r)/R$, [1].

For $R = r$ or ($k = 1$), the corresponding epicycloid is *cardioid* (fig. 5, a) having parametric equations:

$$(\Gamma_{CA}):\begin{cases} x = R(2\cos t - \cos 2t) \\ y = R(2\sin t - \sin 2t). \end{cases} \qquad (4)$$

The entire length of a cardioid is $L = 16r$ and the area covered by the curve is $A = 6\pi r^2$.



a b

Fig. 5 Cardioid (a), nephroid (b)

For $R = 2r$ or ($k = 2$), we get a *nefroid* (fig. 5, b) with parametric equations:

$$(\Gamma_{NF}):\begin{cases} x = r(3\cos t - \cos 3t) \\ y = r(3\sin t - \sin 3t). \end{cases} \qquad (5)$$

4. HYPOCYCLOID

A hypocycloid is the curve described by a point M of a mobile circle rolling over without slipping inside a fixed point.

In order to set the parametric equations of a hypocycloid we chose a reference system with the origin in the centre of the fixed circle and the current point M in the original position M_0, on Ox axis (fig. 6).

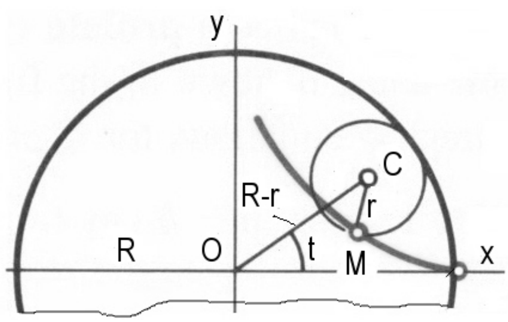


Fig. 6 Hypocycloid

The parametric equations of the common hypocycloid are:

$$(\Gamma_{HC}):\begin{cases} x=(R-r)\cos t+r\cdot\cos\dfrac{R-r}{r}t \\ y=(R-r)\sin t-r\cdot\sin\dfrac{R-r}{r}t. \end{cases} \qquad (6)$$

Based on the location of the point M to the centre of the generating circle we have: the common hipocycloid ($d = r$), shortened hipocycloid ($d < r$) and elongated hipocycloid ($d > r$), where d is the distance from the point M to the center of the mobile circle [1]. These curves are also called *hipotrochoids*.

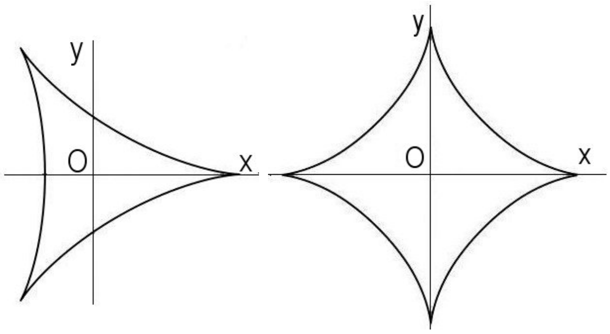
Hypocycloids may have rounded peaks (minimum in relation to the fixed circle), loops or peaks.

If $k = R/r$ is an integer the curve is closed and has k cusps, if k is a rational number p/q, the curve has p cusps, and if k is irrational the curve never closes but fills the space of the great circle except for a circular disk of radius equal to R-2r, located in the center of the large circle.

If $k = R/r$ is an integer, then the hypocycloid has length $L = 8(R-r) = 8R(k-1)/k$, the area between the bow of the hypocycloid and the fixed circle is $A = \pi r^2(3R-2r)/R$, and the area contained inside of the hypocycloid is $A_k = (k-1)(k-2)\pi R^2/k^2$, [1].

In the specific case of $R/r = 3$, an deltoid is obtained (fig. 7, a).

In the specific case of $R/r = 4$, an astroid is obtained (fig. 7, b).



a b

Fig. 7 Deltoid (a), astroid (b)

The parametric equations of an astroid are:

$$(\Gamma_{AT}):\begin{cases} x=R\cdot\cos^3 t \\ y=R\cdot\sin^3 t. \end{cases} \qquad (7)$$

If we remove the parameter t from the equations (7), the Cartesian equation of the astroid is:

$$(\Gamma_{ATc}):\ x^{2/3}+y^{2/3}=R^{2/3}. \qquad (8)$$

An astroid length is $L = 6\,R$, and the area contained within an astroid is $3\pi R^2/8$.

The astroid had a variety of names: tetracuspid or cubocycloid.

In case of the ratio $R/r = 2$, the hypocycloid is reduced to a diameter of the fixed circle.

5. SPHERICAL CYCLOID

A spherical cycloid is a spatial curve described by a point M of a circle that rolls over without slipping on a fixed circle, the two circles are in plans that make an angle with each other constantly ω. Under these geometrical conditions the cycloid are plotted on a sphere (fig. 8, [5]).

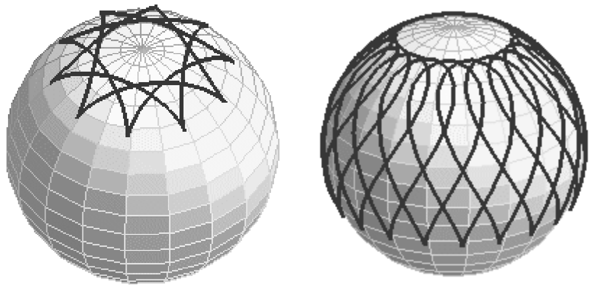
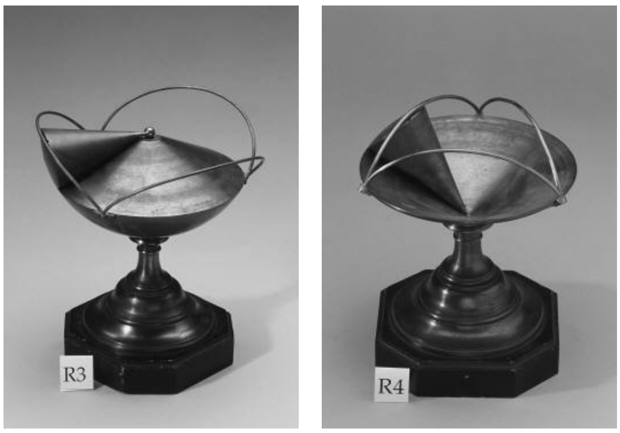


Fig. 8 Spherical cycloids

If $\omega = 0$, we have a hypocycloid and $\omega = \pi$ it is a epicycloid (plane curves).

A spherical cycloid can also be described by a fixed point of a right circular cone as it rolls over without slipping on another fixed rotation cone having the same peak with the first. Under these conditions the cones having a common peak Ω contain the fixed circle and the mobile circle, respectively.

The first cone can run on the convex face or concave face of the fixed cone. Subject to these conditions following cases are defined: the cone running on the convex face of the fixed cone and a sphere-like epicycloid is obtained (fig. 9, a); a cone running on the concave face and a spherical hypocycloid is obtained (fig. 9, b); a cone running on a circle with the peak in the center of the circle and a common spherical cycloid is obtained [2].



a b

Fig. 9 Spherical epicycloid (a), spherical hypocycloid (b)

6. CYCLIC CURVES IN SCIENCE AND TECHNIQUE

From the scientific point of view, a cyclic curve can be *tautochronous*, *brahistochronous* or *isochronous*, properties with applications in physics.

A cycloid (upside down) in a vertical plane is *tautochronous* if a material point on this reaches the lowest point at the same time in any position would start without initial velocity. The tautochronous problem was solved by Christiaan Huygens in 1659.

A cycloid is also *brahistochronous* because it is a curve on which a material point can slide without friction and initial velocity in an uniform gravitational field, so that the displacement time is the lowest in relation to all the curves that unite the two fixed points located at different heights. The brahistochronous problem was solved by Johann Bernoulli, in 1696. This result is considered as a starting point in *variations calculation*.

The *isochronous* problem, in the sense of Huygens, lies in the fact that a material point in a moving periodic without friction, has a period independent of its original position.

In 1673, Christiaan Huygens published in Paris, „*Horologium Oscillatorium*" a very important treaty in the field, in five parts, showing also *a cycloid as a tautochronous curve and isochronous*, and drawings of a *cycloidal pendulum*. If the length of a such pendulum is equal to half the length of a cycloid, then the pendulum suspended from a cuspid of a reversed cycloid so that the thread remains between adjacent cycloid arcs, also describes a cycloidal trajectory due to the geometric property of this curve: a cycloid has also a cycloid as an evolute [4].

The period of a cycloidal pendulum is given by the relationship:

$$T = 2\pi\sqrt{4a/g}\,, \tag{9}$$

where a is the radius of the cycloid generating circle and g is the gravitational acceleration.

In 1687, Isaac Newton (1643–1727) in his work „*Philosophiae naturalis principia mathematica*" demonstrates the equality of a cycloidal pendulum period and that of the oscillations of a little viscous fluid in a U-tube manometer. Mathematically, this demonstration consists of the derivative of the motion equations in a U-tube manometer and setting the period of oscillation.

In cosmology, the Russian physicist Alexander Friedmann (1888–1925) defined, in 1922, the concept called *Finite Universe of Friedmann*, where it is presented a model of non-stationary universe whose radius varies over time depending on its age as a periodic function that can be represented as a cycloid.

These curves meet the fundamental law of gearing, which shows that for the transmission of rotary motion through two toothed gears under constant transmission ratio, tooth profiles must be constructed in such a way that a coomon normal in any point of contact of these profiles to pass a fixed point. For worm gears having cycloidal profile for outdoor gear, the head of the tooth is an epicycloid and the tooth foot is a hypocycloid and for the inner gear the two wheels profile is reversed.

A cycloidal reducer (cycloidal speed reducer or cycloidal drive), built with cycloidal moving surfaces is able to achieve higher ratios in compact size.

For a Wankel rotary engine, the profile of the stator room is a shortened epicycloid with $R = 2r$ (a variant of epitrochoid).

7. CYCLING CURVES IN DESIGN AND ART

A cycloid is used by engineers and designers for designing roller coasters. Being a brahistochronous curve a cycloid allows to obtain highest speeds between the two different heights of the circuit.

In architecture, a cycloid arc was used by Louis Kahn (1901–1974) at Kimbell Art Museum (1966–1972) in Fort Worth, Texas. An American architect named Wallace Kirkman Harrison (1895–1981) used cycloid arcs over the arcades of the main the facades at Hopkins Center for the Arts (1962), Dartmouth College.

With these curves we can make geometric figures having aesthetic impact such as: stained glass windows, mosaics, paving, decorative motifs in tape with repetition or alternation, patterns on ceramic surfaces, stitched or printed patterns on various textile surfaces, curves that generate rotational surfaces, ornamentation, spatial models with aesthetic and didactic character.

8. CONCLUSIONS

Plane and space cycloidal curves apart from the fact that have contributed, through their study, to the evolution of geometric properties of various families of curves were, in many cases, the link between the theory of mathematics and various areas of physics and technology. These curves have many applications in modern technology, in areas which may include: different gears, mechanical transmissions, thermal engines etc. They can be used, with the possibilities of innovation in designing various products. Cycloidal curves through properties and appearance, are widely used in various fields of art, because they are able to solve the aesthetic problems.

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