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A REMARKABLE FIGURE FROM THE BOOK OF LEMMAS

Abstract: *The Proposition 4 from the 11-th volume (called Book of Lemmas) of „Great Books of the Western World" (printed in USA in 1952) presents the properties of a remarkable curve from Antiquity. This curve was studied by Archimedes, who called it Arbelos and it looks like a shoemaker's knife. It consists of 3 semicircles. The first part of the paper is dedicated to the description of an original mechanism, conceived to plot the above-mentioned curve. It is made of a rotating driving element and two dyads of RRR type. The curve plotted by the mechanism checks the equality of areas described by Archimedes. He proved (by using geometric reasonings) that the area of a circle whose diameter is equal to the height of the right triangle inscribed in the Arbelos curve's big semicircle is equal to the Arbelos curve's area. In the second part of the paper, the authors modified the mechanism's parameters and obtained 3 particular cases of Arbelos curves. These particular curves, plotted with the modified mechanism, are also composed of 3 semicircles whilst exhibiting other equalities between areas. The authors used geometric reasonings to prove the equality between the area of each particular curve and the area of a circle whose diameter is equal to the leg of a right triangle inscribed in the big semicircle of the studied particular curve.*

Key words: *Arbelos, curves (particular cases), areas, mechanism.*

1. INTRODUCTION

Encyclopaedia Britannica, Inc. published in 1952 in USA the series" Great Books of the Western World", made of remarkable books, grouped in 54 volumes.

The volume no. 11 is dedicated to the scientists from Antiquity [1]: Euclid – Greak mathematician (325-270 B.C.); Archimedes (287-212 B.C.) mathematician, physicist, astronomer, inventor, military engineer, philosopher, engineer); Apollonius of Perga (Greek surveyor and astronomer 262-190 B.C.); Nicomachus of Gerasa (ancient mathematician and music theorist 60-120 A.C.)

Book of Lemmas attributed to Archimedes by Thābit ibn Qurra is part of the above-mentioned volume. It contains fifteen propositions (lemmas) on circles.

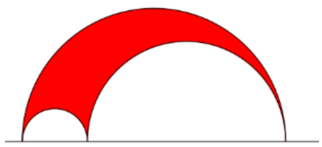
The propositions 4, 5, 6 describe certain properties of a remarkable figure, delimited by 3 semicircles. It is called Arbelos and will be addressed in this paper. In the 14-th lemma, Archimedes introduced another figure (Salinon). For this aim, he applied 10-th Lemma of Euclid's elements from the 2-nd Book.

2. GEOMETRIC ASPECTS OF THE PROBLEM

Arbelos is a geometric figure from Antiquity, studied by Archimedes. The term "arbelos" means cobbler's knife in Greek, Figure 1 [2].

Figure 1 The blade of a knife used by cobblers.

The blade of this knife used by cobblers is similar to the hachured area from Figure 2.



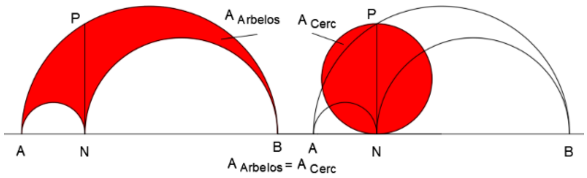


Figure 2 Arbelos.

Archimedes studied the mathematic properties of this figure, figure built from arches of circles. He described it in the Book of Lemmas:

"*Proposition 4: If AB be the diameter of a semicircle and N any point on AB, and if semicircles will be described within the first semicircle and having AN, BN as diameters respectively, the figure included between the circumferences of the three semicircles is "what Archimedes called arbelos"; and its area is equal to the circle on PN as diameter, where PN is perpendicular to AB and meets the original semicircle in P.*" [3]

From Figure 3 it can be observed that:

$$A_1 = A_2 + A_3, \qquad A_2 = A_4 + A_5, \qquad A_3 = A_6 + A_5;$$

$$A_{Arbelos} = A_1 - A_4 - A_6 = \\ = A_2 + A_3 - A_2 + A_5 - A_3 + A_5 = \\ = 2A_5 = A_{Cerc}$$

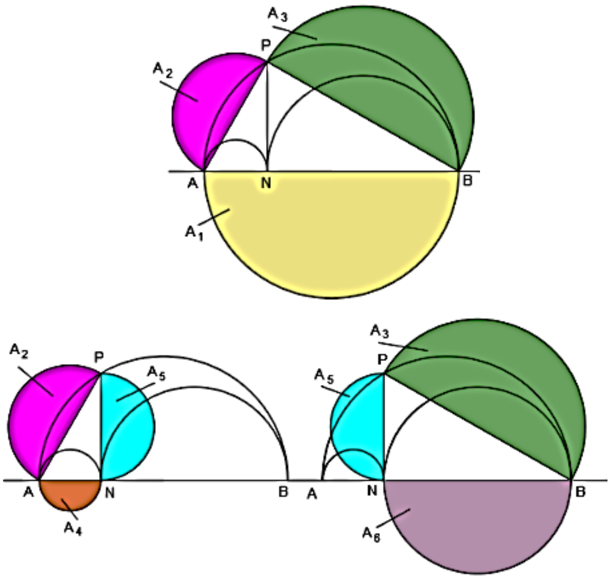


Figure 3 Graphic constructions used to demonstrate that A_Arbelos=A_Cerc

3. TEORETICAL ASPECTS OF THE PROBLEM

• **The built mechanism**

We designed the mechanism from Figure 4, in order to plot Arbelos curves.

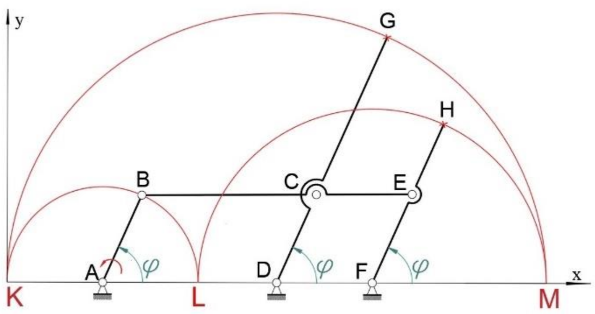


Figure 4 Mechanism used to plot Arbelos curve.

The structural schema is depicted in Figure 5.

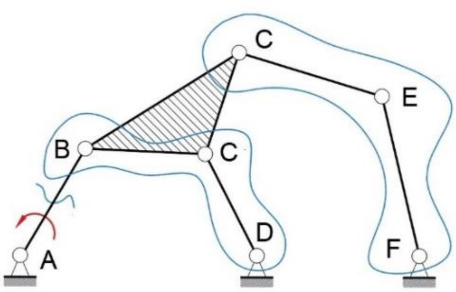


Figure 5 Structural schematic of the mechanism.

The decomposition of the mechanism from Figure 6 yields a leading element AB (with rotating movement) and 2 diades of type RRR: BCD and CEF.

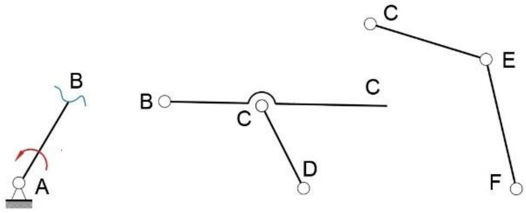


Figure 6 Decomposition of the mechanism.

• **Calculation relations**

The equations (1)...(3) use the following notations:

$$LK = d;\ KM = a;\ ML = d - a;\ AB = \frac{a}{2};$$

$$DG = \frac{d}{2};\ \ FH = \frac{d-a}{2}$$

$$\begin{cases} x_B = \frac{a}{2} + \frac{a}{2}cos\varphi \\ y_B = \frac{a}{2}sin\varphi \end{cases} \tag{1}$$

$$\begin{cases} x_G = \frac{d}{2} + \frac{d}{2}cos\varphi \\ y_G = \frac{d}{2}sin\varphi \end{cases} \tag{2}$$

$$\begin{cases} x_H = a + \frac{d-a}{2} + \frac{d-a}{2}cos\varphi \\ y_H = \frac{d-a}{2}sin\varphi \end{cases} \tag{3}$$

• **Graphic results**

Figure 7 depicts an Arbelos example, plotted with the mechanism from fig. 4, for which the mathematical properties stated by Archimedes in Proposition 4 from the Book of Lemmas are verified.

In this case we consider d=100 and a=30.

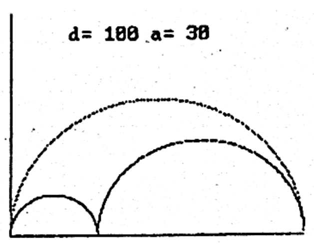


Figure 7 Arbelos plotted with the generator mechanism, d=100, a=30.

The tree curves below are considered as particular cases for Arbelos. They will be referred as A_{CP}. It will be demonstrated that the area of A_{CP} is equal to the area of a circle, denoted by A_C, whose diameter is equal to the chord MN, as revealed by Figures 9, 12 and 15.

4. PARTICULAR CASES OF ARBELOS CURVES

• **First Arbelos particular case**

Figure 8 was obtained by modifying the mechanism dimensions such as $d < a$, with $d = 30;\ a = 70$. The semicircle with the diameter ML is placed on the other side of the semicircles whose diameters are KM and KL respectively (Figure 9).

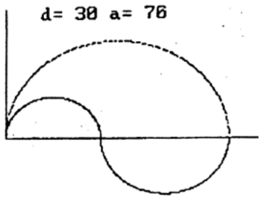


Figure 8 Curve plotted with the generator mechanism for d=30 and a=70.

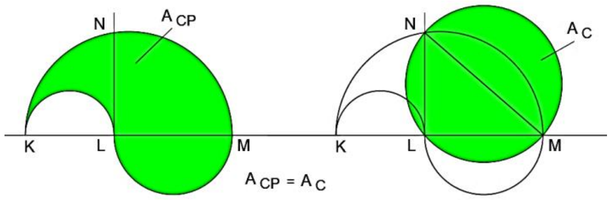


Figure 9 $A_{CP} = A_C$ for d=30 and a=70.

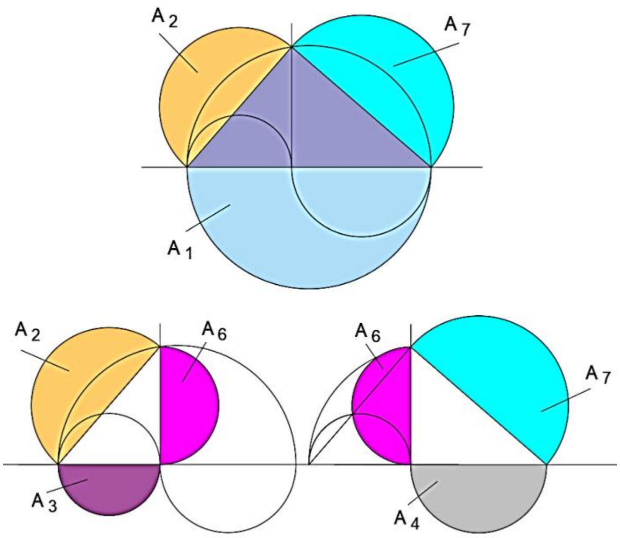


Figure 10 Graphic constructions used to demonstrate that $A_{CP} = A_C$, for $d < a$.

It can be observed that (Figure 10):

$$A_1 = A_2 + A_7, \qquad A_2 = A_3 + A_6, \qquad A_7 = A_6 + A_4$$

$$A_{CP} = A_1 - A_3 + A_4 = A_2 + A_7 - A_2 + A_6 + A_7 - A_6$$
$$= 2A_7 = A_C$$

• Second Arbelos particular case

Figure 11 depicts the curve plotted with the mechanism for $d < 0 \; and \; Abs(d) < a$.

In this case d= - 30 and a=70 (Figure 12).

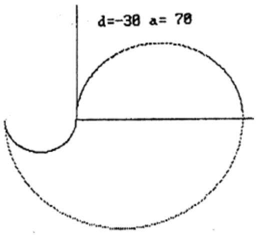


Figure 11 Curve plotted with the generator mechanism for d= - 30 and a=70.

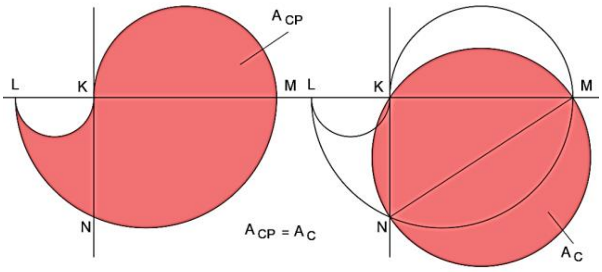


Figure 12 $A_{CP} = A_C$ for d= - 30 and a=70.

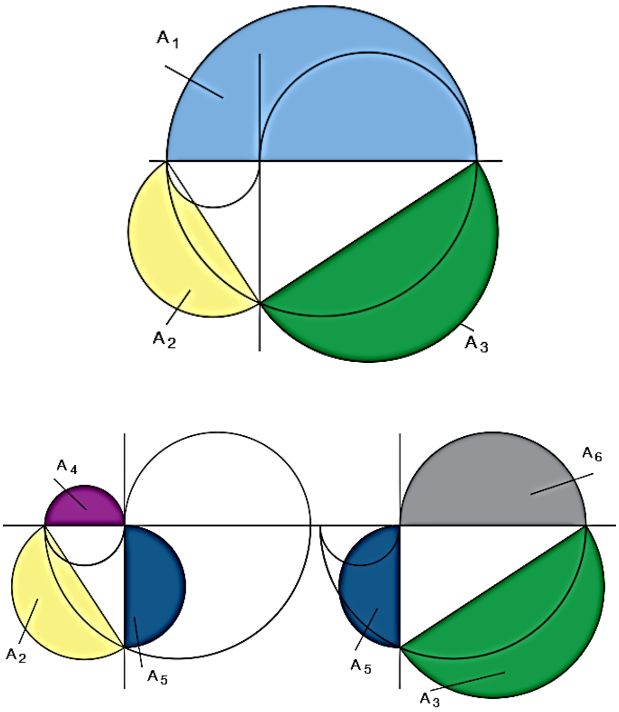


Figure 13 Graphic constructions used to demonstrate that $A_{CP} = A_C$, for $d < 0 \; and \; Abs(d) < a$.

It can be observed that (Figure 13):

$$A_1 = A_2 + A_3, \qquad A_2 = A_4 + A_5, \qquad A_3 = A_6 + A_5$$
$$A_{CP} = A_1 - A_4 + A_6 =$$
$$= A_2 + A_3 - A_2 + A_5 + A_3 - A_5 =$$
$$= 2A_3 = A_C$$

• Third Arbelos particular case

Figure 14 depicts the curve plotted with the mechanism for $a < 0 \; and \; Abs(a) > d$.

In this case d=30 and a= - 70 (Figure 15).

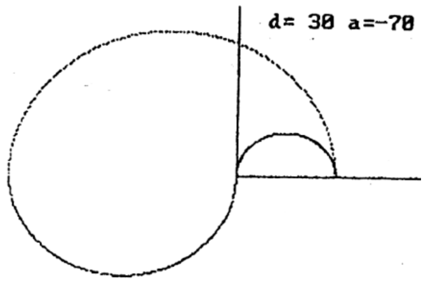


Figure 14 Curve plotted with the generator mechanism for d= 30 and a= - 70.

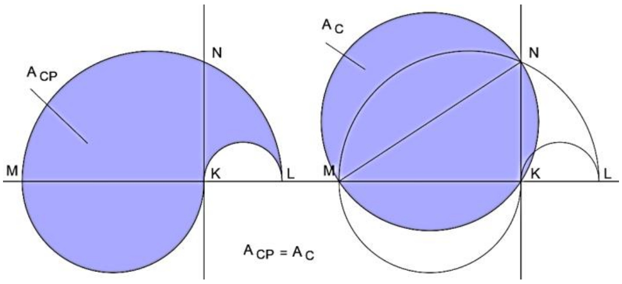


Figure 15 $A_{CP} = A_C$ for d= 30 and a= - 70.

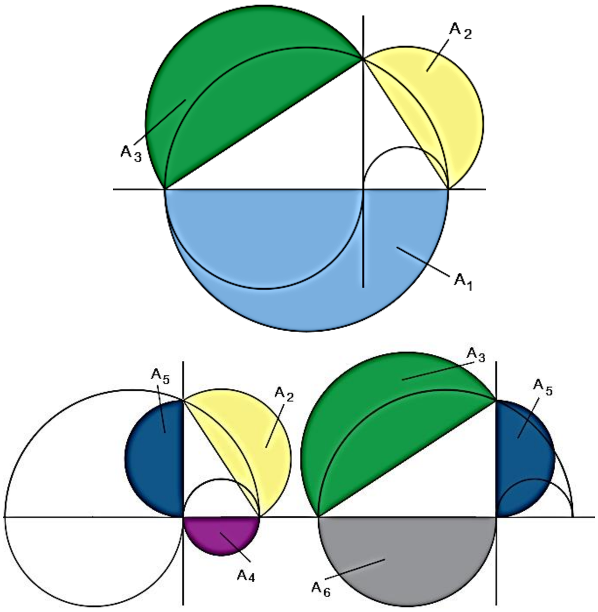


Figure 16 Graphic constructions used to demonstrate that $A_{CP} = A_C$, for $a < 0$ and $Abs(a) > d$.

It can be observed that (Figure 16):

$$A_1 = A_2 + A_3, \quad A_2 = A_4 + A_5, \quad A_3 = A_6 + A_5;$$
$$A_{CP} = A_1 - A_4 + A_6 =$$
$$= A_2 + A_3 - A_2 + A_5 + A_3 - A_5 =$$
$$= 2A_3 = A_C$$

5. CONCLUSIONS

- Archimedes studied the mathematic properties of a figure built from arches of circles, similar to the blades of knifes used by cobblers. It is called Arbelos and is described in Book of Lemmas.
- A mechanism able to generate this curve was built.
- Particular cases of the Arbelos curve were created with this mechanism.
- It was demonstrated that these particular cases have an area equal to the area of a circle ($A_{CP} = A_C$).

REFERENCES

[1] https://en.wikipedia.org/wiki/Great_Books_of_the_Western_World. *Accessed*: 2022-01-15.
[2] https://pballew.blogspot.com/2020/10/the-shoemakers -knife-cuts-beautiful.html, *Accessed*: 2022-01-15.
[3] https://www.cut-the-knot.org/ Curriculum/ Geometry/ BookOfLemmas/ index.shtml, *Accessed*: 2022-01-15.

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