Iulian POPESCU, Ludmila SASS, Alina DUTA, Marcela POPESCU, Alina Elena ROMANESCU

DOUBLE-HEART CURVE GENERATED BY AN ORIGINAL MECHANISM

Abstract: *Different variants of cardioids were reviewed and authors decided to an end to focus on the usability of a Double-heart curve. Starting from the geometric information related to its generation, the synthesis of an original generating mechanism was performed firstly, followed by the realization of its structural analysis. Mechanism's positions were computed and the imposed curve was generated. The variations of certain trajectories were deduced from the corresponding diagrams. The mechanism sizes were modified and a wide class of curves presenting interest from the geometric point of view was obtained. Some mathematical properties of the curves generated by the above mechanism are studied (binormal vector, tangent equation in a current point, normal plane equation, tangent versor, versor of normal to the curve).*

Key words: *Double-heart curve, curve generator mechanism.*

1. INTRODUCTION

The cardioids were intensively studied. The authors of [1] were concerned with drawing cardioids as particular representations of epicycloids. Modalities to trace a cardioid as an epicycloids are presented in [2]. In the same paper the cardioid's rafter, which is similar to the cardioid, additionally including a loop around the returning point. Several types of cardioids, along with their equations, are provided in [3], concluding that several variants are available. The cardioid generation as a cyclic curve is presented in [4].

A domain for the utilization of cardioids is presented through examples in [5], considering aesthetic models with metallic fences and metal screens for windows.

The Double-heart curve, studied in 1647 by Grégoire de Saint Vincent and in 1750 by Cramerin respectively is approached in [6]. The equation of the curve (which is quartic), is provided by using Cartesian coordinates, mentioning that it is a polyzomal curve. It was studied by Cayley in 1868. The geometrical generation of the double-heart curve is presented in [6]. The animation of a line moving such as to preserve its parallelism to the abscise is employed. The line's ends are moving along two tangent circles, one being placed inside the other one and the line's middle point traces the above mentioned curve (fig. 1).

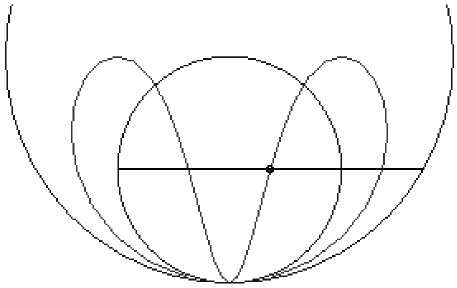

Fig. 1 The line's middle point traces the Double-heart curve. [6]

This information was used to realize the synthesis of the generating mechanism.

2. SINTHESIS OF MECHANISM

One considered the inner circle as having the center denoted by the point A and a diameter equal to the radius of the outer circle, whose center is in the point H (fig. 2). The point B rotates along the inner circle, whilst the rod BC whose length is variable, has to preserve its parallelism to the abscise.

Therefore coulisses were placed in the point C. They are perpendicular one over the other and welded. The coulisse from F preserves the parallelism of the coulisses from C with the axis x and y. The element 4 is linked through a rotation couple to the element 3. The points A and H have the same ordinate.

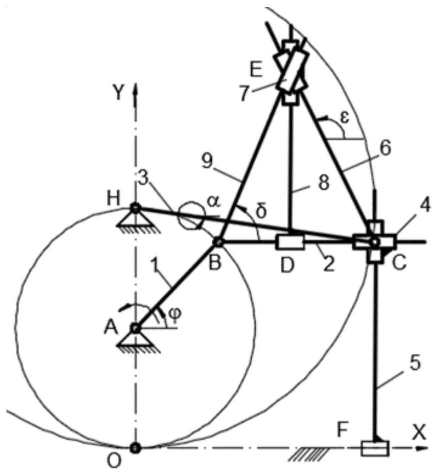

Fig. 2 The synthesised mechanism.

The difficult problem consists in determining the point D within the mechanism, as long as it is placed in the middle of the rod BC which has a variable length.

Fig. 3 depicts the adopted technique – drawing the perpendicular line on the middle of the BC segment, according to the method from the geometric drawing. The perpendicular line is crossing through the points E and D. Point D is placed in the middle of the BC segment.

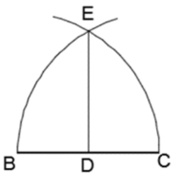

Fig. 3 Determining the middle of the BC rod.

The kinematic chain relying on fig. 3 is depicted in fig. 4. The coulisse D slides along BC. The elements BE and CE have equal lengths. The coulisse from E allows for the sliding of the element 6 when the length of BC is changing.

3. MECHANISM STRUCTURE

The mechanism has 9 mobile elements and 13 couples of 5-the class, as follows: A(0,1); B(1,2); B(1,9); D(2,8); H(3,0); C(2,4); C(4,5); C(4,6); C(3,4); F(5,0); E(6,7); E(8,7); E(9,7). The degree of mobility is: M=3n-2C5-C4 =3.9-2.13=1.

Fig. 4 depicts the structural schematic of the mechanism and its decomposition in kinematic groups. A leading element and two triads are obtained (fig. 5).

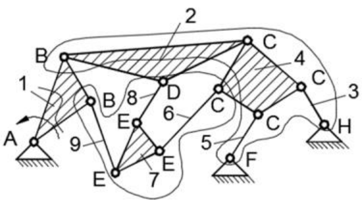

Fig. 4 The structural schematic of the mechanism.

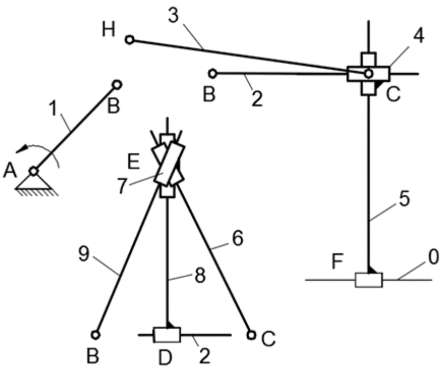

Fig. 5 A leading element and two triads.

4. RELATIONS AND GRAPHICAL RESULTS

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Based on fig. 2 we obtained:

$$\begin{cases} x_B = x_A + AB.\cos\varphi \\ y_B = y_A + AB.\sin\varphi \end{cases} \quad (1)$$

$$\begin{cases} x_C = x_B + BC.\cos 0^0 = x_H + HC.\cos\alpha \\ y_C = y_B = y_H + HC.\sin\alpha \end{cases} \quad (2)$$

$$\begin{cases} x_D = x_B + 0{,}5BC.\cos 0^0 \\ y_D = y_B \end{cases} \quad (3)$$

Eq. (1) yields the coordinates of B, and the equation (2) are used to get α and BC.

The coordinates of D (the point tracing the curve) are obtained from (3). The triangle BEC is isoscel (as revealed by fig. 3). It means that BE=EC, and the angles EBC and ECB are equal.

The following initial data were considered, considering millimeters as units of measure:

Fig. 6 depicts the plotted mechanism in a certain position.

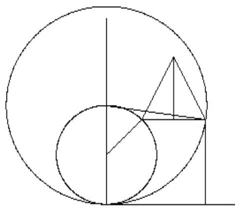

Fig. 6. The plotted mechanism in a certain position considering the sign (+).

sin α is yielded by equations (2), and for next, cos α, one needs to extract a square root with two possible signes ($\pm$). Two solutions are therefore possible.

From the constructive point of view, the mechanism is initially positioned with CF in the right side of A (corresponding to the sign (+)), according to fig. 6. Fig. 7 depicts the solution corresponding to the sign (-).

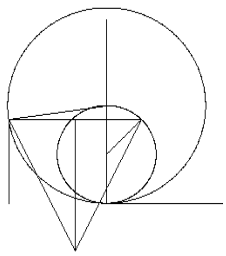

Fig. 7 The mechanism in a position for the sign (-).

The successive positions for the mechanism are depicted in fig. 8 (for the sign (+)) and in fig. 9 for the sign (-).

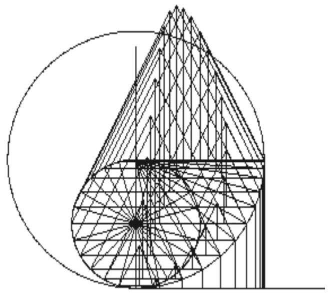

Fig. 8 Successive positions for the mechanism considering the sign (+).

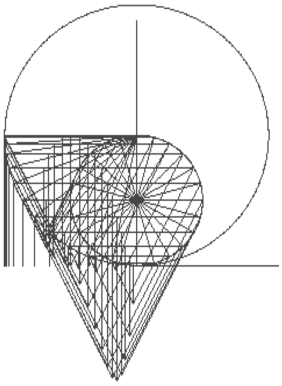

Fig. 9 Successive positions for the mechanism considering the sign (-).

Fig. 10 depicts the side BC with variable length for the solution with (+) whilst fig. 11 corresponds to the sign (–).

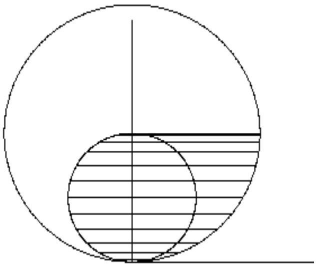

Fig. 10 Variation of the side BC considering the sign (+).

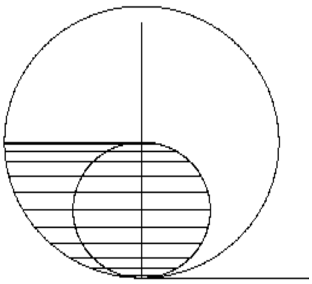

Fig. 11 Variation of the side BC considering the sign (-).

Fig. 12 presents the curve's side considering the sign (+) whilst fig. 13 is dedicated to the counterpart situation. The final curve is depicted in fig. 14, being similar to that from fig. 1.

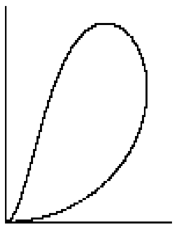

Fig. 12 Curve's side considering the sign (+).

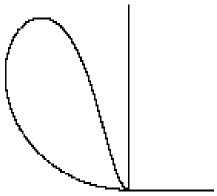

Fig. 13 Curve's side considering the sign (-).

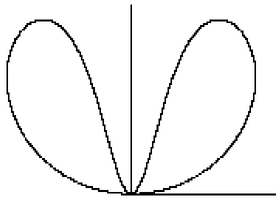

Fig. 14 Double-heart curve generated by the mechanism.

5. MATHEMATICAL PROPERTIES OF THE GENERATED CURVES [7]

Considering the actual selection for the reper placed on the *xOy* plane, one selected a parameter (denoted by *t* in fig. 15) consisting in the angle φ formed by the vectorial radius AB with the parallel to *Ox* crossing through A. With this notation:

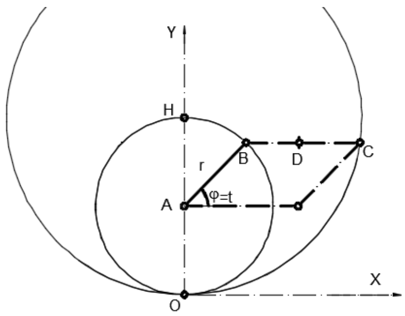

Fig. 15 Analytical determination of the coordinates of point D in the xOy reper.

- The point B has the coordinates:

$$\begin{cases} x_B = r.\cos t \\ y_B = r + r.\sin t \end{cases} t \in [0,2\pi] \quad (4)$$

- The equation associated to the line BC, parallel to the axis *Ox*:

$$y = r + r.\sin t \quad (5)$$

- The big circle, with the radius *R=2r* and the center *H(0,2r)* has the equation:

$$\mathcal{C}: x^2 + (y - 2r)^2 = 4r^2 \qquad (6)$$

- We consider that:

$$C = \mathcal{C} \cap (BC) \qquad (7)$$

then the coordinates of the point C can be deduced from the system of equations:

$$\begin{cases} x^2 + (y-2r)^2 = 4r^2 \\ y = r + r\sin t \end{cases} \quad (8)$$

By substituting y in the 1-st equation, we get:

$$x_C = \pm\sqrt{(1+\sin t)(3-\sin t)} \quad (9)$$

For:

$$x_C = +r\sqrt{(1+\sin t)(3-\sin t)} \quad (10)$$

$$\begin{cases} x_D = \dfrac{x_B + x_C}{2} \\ y_D = y_B = y_C \end{cases} \quad (11)$$

the parametrical equations of the curve, corresponding to the first loop, which is placed in the quarter, can be written as:

$$\begin{cases} x_D{}^1 = \dfrac{r}{2}\left(\cos t + \sqrt{(1+\sin t)(3-\sin t)}\right) \\ y_D{}^1 = r(1+\sin t) \end{cases} t \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \quad (12)$$

One can notice that it corresponds to the first positioning of the mechanism (with CF placed at the right of A, fig. 6).

For:

$$x_C = -r\sqrt{(1+\sin t)(3-\sin t)} \quad (13)$$

following the same steps, the parametric equations of the curve, corresponding to the 2-nd loop, placed in the 2-nd quadrant, are:

$$\begin{cases} x_D{}^2 = -x_D{}^1 \\ y_D{}^2 = y_D{}^1 \end{cases} t \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \quad (14)$$

One should remark that it corresponds to the second positioning of the mechanism (with CF placed at the right of A, fig. 7).

5.1. Remark 1

By selecting another coordinate reference, with the origin in point A and following identical steps, one gets the parametric equations in which the expression of x_D is the same, by the expression of y_D is simpler.

It means that:

$$y_D = r.\sin t \quad (15)$$

5.2 Remark 2

As $(\forall)t \in R: -1 \le \sin t \le 1$, so $0 \le 1 + \sin t \le 2$, it means that $0 \le r(1+\sin t) \le 2r$, meaning $0 \le y_D \le 2r$; in other words, the maximum value of y_D is 2r. For the first loop, this value is reached when $t = \pi/2$ and in this case $x_D = r$, which is the maximum of the first loop.

$$V_1 = \left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) = (r, 2r) \quad (16)$$

In a similar manner, the loop of the second loop is:

$$V_2 = (-r, 2r) \quad (17)$$

Obviously, for $t = -\pi/2$ and $t = -3\pi/2$, one gest the point O(0,0) which can be found on both loops (being the only common point).

5.3 Remark 3

Let us consider the graphical representation of the curve as being a function $x \to y = f(x)$. When the first loop is considered,

- For $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, f is strictly increasing
- For $t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, f is strictly decreasing

Similar remarks can be made relative to the second loop.

5.4 Remark 4

Considering the selection made for the system of the coordinates, the equation (14) reveals the symmetry of curve with respect to the axis Oy. In any other system of coordinates, the curve is symmetric relative to the mediator of the segment $[V_1 V_2]$, where V_1 and V_2 are the loops peaks.

The use of Frenét's trihedral provides more information than any other fix or mobile Cartesian reper. As long as we deal with a plane curve, the osculator plane is actually the plane of the curve (xOy).

The deviation of the curve from the osculator plane is the torsion which is null (as it always is for a plane curve).

5.5 Binormal vector

The binormal vector is perpendicular to the osculator plane in the current point of the curve. It means that is perpendicular to (xOy) in the current point of the curve. It actually represents the line for which $\bar{a}(0,0,1)$ is playing the role of leading vector and which is crossing through the point $M(x_D, y_D, 0)$, with x_D *and* y_D representing the coordinates of the point D from the curve:

$$\begin{cases} x = x_D \\ y = y_D, h \in R \\ z = h \end{cases} \quad (18)$$

The set of all binormal vectors generates a cylindrical surface for which the given curve plays the role of leading curve.

5.6 Tangent equation in a current point

Because:

$$\begin{cases} x'(t) = \frac{r}{2}\left(-\sin t + \frac{\cos t - \sin t \cos t}{\sqrt{(1+\sin t)(3-\sin t)}}\right) \\ y'(t) = r\cos t \end{cases} \quad (19)$$

$$t \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right), \text{ because } t \neq -\frac{\pi}{2}, t \neq \frac{3\pi}{2}$$

the tangent equation in a certain point (x_D, y_D) (where $t_0 \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is fixed but with no other specifications), is:

$$\frac{x - x_D}{x'(t)} = \frac{y - y_D}{y'(t)} \quad (20)$$

In parametric expression:

$$\begin{cases} x = \frac{r}{2}\left(\cos t_0 + \sqrt{(1+\sin t_0)(3-\sin t_0)}\right) + \\ + s.\frac{r}{2}\left(-\sin t_0 + \frac{\cos t_0 - \sin t_0 \cos t_0}{\sqrt{(1+\sin t_0)(3-\sin t\ )}}\right), s \in R \\ y = r(1+\sin t_0) + s.r\cos t_0 \end{cases} \quad (21)$$

5.7 Normal plane equation

The normal plane is determined by the current point and has as leading vector the leading vector of the tangent to the current point in the curve. It is described by equation:

$$x'(t).(x - x_D) + y'(t).(y - y_D) = 0 \quad (22)$$

5.8. Tangent versor

Denoting by:

$$\sqrt{(1+\sin t)(3-\sin t)} = E(t) \quad (23)$$

Then:

$$\begin{aligned} &\sqrt{(x'(t))^2 + (y't))^2} = \\ &= \sqrt{\frac{r^2}{4} \cdot \frac{(-\sin t.E(t) + \cos t - \sin t\cos t)^2}{(E(t))^2} + r^2\cos^2 t} = \\ &= \frac{r}{2E(t)} . \sqrt{(E(t))^2(1+3\cos^2 t) - 2\sin t\cos E(t).} \\ &\overline{.(1-\sin t) + \cos^2 t(1+\sin t)^2} \stackrel{NOTATIE}{=} \frac{r}{2}F(t) \end{aligned} \quad (24)$$

It comes out that the versor of the tangent $\overline{\tau}(\tau_1(t), \tau_2(t))$ has the components, for $t \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$:

$$\begin{cases} \tau_1(t) = \frac{x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} = \\ = \frac{-\sin t.E(t) + \cos t - \sin t\cos t}{F(t)} \\ \tau_2(t) = \frac{y'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} = \frac{2\cos t}{F(t)} \end{cases} \quad (25)$$

5.9 Versor of normal to the curve

The versor of normal to the curve in the current point is $\overline{n}(n_1, n_2)$, where $n_1 = \tau_2(t)$ and $n_2 = -\tau_1(t)$ which means that: $\overline{n}(\tau_2(t), -\tau_1(t))$

5.10 Equation of the normal to the curve in the current point

The equation of the normal to the curve in the current point $(x_D, y_D), t_0 \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$, is

$$\frac{x - x_D}{y'(t)} = \frac{y - y_D}{-x''(t)} \quad (26)$$

The parametric representation is:

$$\begin{cases} x = \frac{r}{2}\left(\cos t_0 + \sqrt{(1+\sin t_0)(3-\sin t_0)}\right) + \\ + s.r\cos t_0 \\ y = r(1+\sin t_0) + \\ + s.\frac{r}{2}\left(\sin t_0 - \frac{\cos t_0 - \sin t_0\cos t_0}{\sqrt{(1+\sin t_0)(3-\sin t_0)}}\right) \end{cases}, s \in R \quad (27)$$

5.11 Equation of the rectifying plane

The rectifying plane is determined by the current point and its normal vector is the leading vector of the normal to the curve in the current point. Its equation is:

$$y'(t)(x - x_D) - x'(t)(y - y_D) = 0 \quad (28)$$

5.12 (Plane) curve curvature

The (plane) curve curvature is:

$$K(t_0) = \frac{|x'(t_0).y''(t_0) - x''.y'(t_0)|}{\left(\sqrt{(x'(t_0))^2 + (y'(t_0))^2}\right)^3} \quad (29)$$

where, for $t \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$:

$$\begin{cases} x''(t) = -\frac{r}{2}(\cos t + \frac{(\sin t + \cos^2 t - \sin^2 t)E^2(t)}{E^3(t)} + \\ + \frac{(\cos t - \sin t\cos t)^2}{E^3(t)}) \\ y''(t) = -r\sin t \end{cases} \quad (30)$$

5.13 Remark 5

If the small circle, whose radius is r, is not tangent from inside to the big circle (whose radius is 2r) in the origin O, but it is tangent from inside in the point of coordinates (0,4r) (diametrically opposed), than one gets a curve symmetric relative to the line described by the equation y=2r. This last one is crossing through H and is parallel to Ox, as in fig. 16.

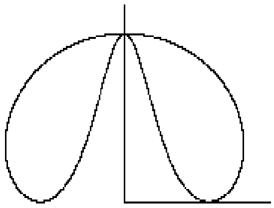

Fig. 16 The curve when the small circle is tangent from inside in the point of coordinates (0,4r).

5.14 Remark 6

The curve is the symmetric relative to *Oy*. Therefore, through rotation by an angle between 0 and π it can generate a compact, bounded and closed surface. The surface is smooth (except for the point O) and connected through arcs. The body of rotation bounded by this surface is also compact and connected through arcs, but it is not convex.

5.15 Remark 7

If the small circle with radius r is tangent from inside in a point to a bigger circle with radius

$R = k.r, k > 1$, then all the above can be adapted by analogy. One can realize an interesting qualitative, quantitative and comparative study relative to the obtained curves, for $k = 1{,}5; k = 2; k = 2{,}5; k = 3, etc.$

6. CONCLUSIONS

The paper provides a contribution to the analysis and synthesis stages of the mechanisms used to generate plane curves.

The starting point consists in mechanisms used to plot cardioids, considering details on the shape of the Double-heart curve.

The animation of the geometrical genertion of this curve was used to achieve the synthesis of the generating mechanism consisting in 9 mobile elements and 13 couples of class V.

A complicated analysis was performed: the obtaining of a kinematic chain in which the middle of an element with variable length represent the generating element of the studied curve.

The mechanism's structure was studied and afterward the relations based on the contour's method were written. Their successive positions were determined, which finally yielded the imposed curve.

Original particular curves of various shapes (some of the similar to the imposed curve) were obtained by modifying the lenghts of certain sides of the mechanism.

The mathematic study of the Double-Heart curve was also performed.

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Authors:

Professor Iulian POPESCU, Faculty of Mechanical Engineering, University of Craiova, Romania, rodicaipopescu@yahoo.com

Associate Professor PhD Ludmila SASS, Faculty of Mechanical Engineering, University of Craiova, Romania, ludmila_sass@yahoo.com

Associate Professor PhD Alina DUTA, Faculty of Mechanical Engineering, University of Craiova, Romania, duta_alina@yahoo.com

Associate Professor PhD Marcela POPESCU, Faculty of Science, University of Craiova, Romania, marcelacpopescu@yahoo.com

Lecturer PhD Alina Elena ROMANESCU, Faculty of Mechanical Engineering, University of Craiova, Romania, romanescualinaelena@yahoo.com