Marija OBRADOVIC

GEOMETRIC REDESIGN OF THE SUBDIVIDED SURFACE OF CC II: APPLICATION IN ARCHITECTURE

Abstract: *The concave polyhedral surface of CC II can be used as a structural template for architectural design of domes, roofs or other covering or stand-alone structures. Subdivision of CC II faces in geometrically defined way can be observed as a part of the design process, if done with the intention to contribute to the aesthetic quality of the building itself. In this paper we discuss certain interventions on the tiled triangular faces of the CC II by regular triangles and hexagons, in order to get patterns applicable in architecture. By using different colors and / or materials of the tiles, we can get solutions that add the decorative layer to the structural one. Examining various solutions, this research focused on the D_3 subdivision of the lateral polyhedral triangle (LPT) and in the resulting uniform tilings, searching for the ways to overcome monotony of highly symmetrical patterns. As opposed to exploring the ways of assembling the tiled LPTs with assigned layouts of tiles into shape of the CC II in order to get desired patterns on its surface, this paper explores the creation of various patterns within the existing, formerly obtained uniform tilings (2-uniform, trihexagonal tiling). A couple of conceptual solutions are given, as an illustration of the idea.*

Key words: *Concave cupola, tiling, subdivision, triangle, hexagon, architecture, pattern.*

1. INTRODUCTION

The geometric problem of tiling a planar surface is closely related to architectural applications, because it has been brought out of the practical problem: how to pave flat surfaces such as floors, roofs, walls, and the like. The earliest records of tilings can be found in Ancient Mesopotamia [26], but we can encounter this subject throughout the history of architecture.

Out of 2D space, the problem can be transferred to 3D, so we can search for solutions which imply tiling polyhedral surfaces. These surfaces can also be tiled if their subdivided (tiled) planar net corresponds to Euclidean tiling. Since the number of such solutions can be infinite, depending on the chosen tiles' arrangement method, as well as on the number of subdivisions of the polyhedral faces themselves, geometricians, architects and other scientist who have been engaged in this problem have mostly confined themselves to k-uniform tilings. Magnus Wenninger [28], Buckminster Fuller [10], Keith Critchlow [8], Branko Grünbaum [14]... are some of the most important names that dealt with this subject in their work. Due to an unlimited number of k-uniform tilings alone [11], not to mention the variations within the already observed ones, there is a plenty of room for research, and especially for applications in this field.

In this paper we will focus on the deltahedral surfaces of the *concave cupolae of the second sort* (CC II) [24], [20], [21], [22], [23] and the possibilities for such structures to be used in architecture. The aforementioned solids (CC II hereinafter) are polyhedra whose properties and metric relations have been elaborated in previous studies [24], [20]. There are 14 representatives of CC II: 7 of CC II-M (upper row on Fig. 1) and 7 of CC II-m (lower row on Fig. 1), over the same ($4 \leq n \leq 10$) bases.

Such spatial structures can be used as a template for architectural elements such as domes (as the origin of the word "cupola" suggests), but also for stand-alone objects. Their fragments can also be used as parts of objects, roof surfaces, wall surfaces, etc., especially suitable whenever regular polygonal bases ($4 \leq n \leq 10$; $8 \leq 2n \leq 20$) are concerned [25], including bases such as heptagon or nonagon.

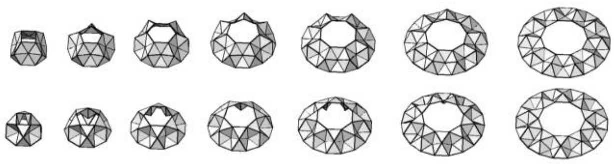

Fig. 1 Fourteen representatives of CC II. [20], [25]

The need to subdivide the triangular lateral faces of these polyhedra (or lateral polyhedral triangle, LPT hereinafter) can be pointed out in several situations: when the LPT of larger dimensions should be split because of the transport or static characteristics of the applied material, as well as in situations where certain visual or design effects are desired. This study is based primarily on the latter, giving some ideas that can be applied not only on the actual case of CC II, but also on any other deltahedral surface.

2. APPLICATION OF CC II IN ARCHITECTURE

In several previous studies [24], [22], [23], [21] and [18], CC II have been examined in terms of suitability for application in architecture. Their feasibility has been shown, static stability proven, and in [24], [21], [23] and [25] some ideas for architectural design has been given.

Due to the equilateral triangles in its composition, their congruence and their simple shape suitable for serial production, the form of the CC II can be easily assembled on the site. Also, since any CC II can be completed by folding the one-piece planar net, it provides additional possibilities for application in different building systems. Thus, their construction can be quite different[1], but we will focus on the conceptual design provided by tiling the LPT of CC II.

In structural terms, the shape of CC II is advantageous because the positions of the vertices are geometrically conditioned [24]. Therefore, it is stable and suitable for larger spans, with no need for additional support elements. However, if we perform such a structure in larger dimensions, the triangular faces themselves might be extensive and difficult for handling, so we need to subdivide them.

For example, to cover a decagonal base of span *R=20m*, we choose the shape of CC II-5. The edge of the polyhedron will be ≈ *6.50m*, so the surface of LPT equals *18.29m²*. These dimensions might be too large to cover in some materials (e.g. glass) as a whole, so triangular faces will have to be dissected. If we adopt the edge subdivision into $3 \leq b \leq 9$ sections, as in [25], the edge length of thus obtained tiles ranges from a_3*=2.17m*, to a_9*=0,72m*, which is convenient for prefabrication and on-site handling. As the spans vary, the dimensions of the tiles will proportionally change.

The subdivision of the triangular faces can also be made solely for the purpose of the design's visual impact, on which the accent will be put in this paper.

3. THE PHASES OF THE RESEARCH

Subdivision of triangular faces of CC II can be done in different ways, from an arbitrary division, to a division into regular polygons. The latter one corresponds to the problem of the Euclidean tilings. This problem has been thoroughly researched by several authors, starting form Kepler [16], so we only mention the most significant: Coxeter [7], Grünbaum [14], Critchlow [8], Wenninger [28], Ghyka [12], Conway [5], [6], Chavey [2], [3].

Being related to the tiling problem, we can adopt the matching triangular segment of a k-uniform Euclidean tiling, as a solution of the problem at hand, as shown in [25]. Thereby we chose a tiling that includes only equilateral triangles and regular hexagons as tiles, because no other shapes can arise within the uniform subdivision of the initial equilateral triangle (LPT) [15].

However, segments of the k-uniform tilings will not be the best way of solving such a problem, because it would require a lot of painstaking research in comparing, fitting, deciding, accepting or rejecting the tiling segments as solutions, as many of them would not be valid (would not fit into the triangular frame without residues or overlaps). We can say that this "gestalt" method is based more on intuition and visual perception, than on an exact, targeted solution. So, if we would like to chose from a number of valid solutions, then we will take the reverse steps: we will divide the triangle (LPT) into triangles and hexagons, put them back in the net of the cupola (Fig. 2) and then choose, between many solutions, the most suitable one for technical or aesthetic reasons.

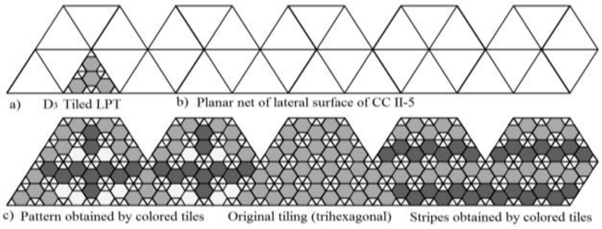

Fig. 2 a, b) Tiled LPT set in the segment of the CC II's net, c) patterns obtained by recoloring certain tiles in the uniform tiling (trihexagonal tiling) in the CC II net.

At this stage of the research, we are no longer satisfied with uniform tilings obtained by mere alignment of subdivided LPT in the net of CC II and consequent patterns that is created on its surface. Hence, within the previously obtained tiling, we modify the solution by changing color, material, relief of certain tiles, or by their omission. In this way, in the faster and more precise way we get the desired pattern, without deliberation about what sort of LPT subdivision, or what kind of LPT orientation within the CC II's net we need to get the desired solution (Fig. 2 c).

4. TRIANGLE FREQUENCY AND THE ARRANGEMENT OF TILES

In order to divide the LPT into another (smaller) triangles and regular hexagons, we will use an auxiliary triangular grid (Fig. 3 b) within the triangle [13], which we obtain by dividing the edges of the triangle into ***b*** [4], [25] sections (Fig. 3 b), and then connect these new vertices by the parallel lines. This is a procedure that corresponds to determining the frequency of the principal polyhedral triangle (PPT) in geodesic polyhedra [13], [4], [10]. The frequency is denoted by ***b***, as in this case it is equal to the number of edge segments. We will use this division in order to fit hexagons within such a triangular grid (Fig. 3 c) and then look for possible solutions.

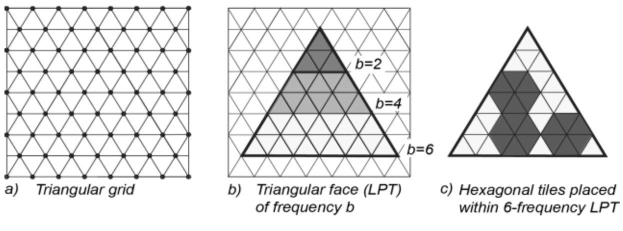

Fig. 3 a)Triangular grid b) Uniform subdivision of LPT and the frequency of equilateral triangle, c) Placing regular hexagons within triangular grid of subdivided LPT of frequency 6. [25]

For example, to illustrate the procedure, this study holds down to the division - frequency of 3<***b***<9 [25]. The higher the frequency of the triangle, the number of hexagons within the triangle will be greater. The higher

[1] from a low budget, provisional construction that can be performed by a one-man constructor, via classical roofing system, to highly developed technology (including precast flat panel system, diagrid system, shell plates etc.…) where the contracting teams are involved.

the number of hexagons we can place within the triangle, the number of their different layouts will be greater, as we can see in Fig. 4, for 6-frequency LPT.

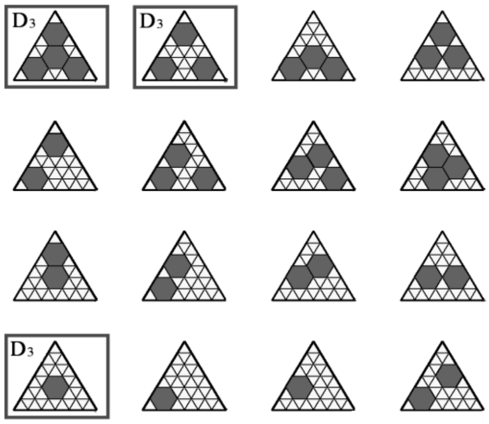

Fig. 4 The example of 6-frequency triangle (LPT) with distinct arrangements of hexagonal tiles and highlighted cases with D_3 symmetry.

Determination of the number of all the possible tiles' dispositions for the observed ***b***-frequency, requires engaging in more detailed combinatorial calculations, which this study will not deal with.

It should be mentioned that among these solutions we can find the ones where hexagonal tiles:

- are separated, without contact
- touch by the edges
- touch by the vertices
- touch both by the edges and vertices
- some hexagonal tiles touch, and others are separated,

as well as those where hexagonal tiles are arranged with different levels of symmetry: asymmetric, plane symmetric, rotationally symmetric, centrally symmetric and D_3 symmetric - where all the mentioned symmetries are involved. This is important because of the continuity of the joined tiles, helping patterns to be defined more clearly.

In the previous research [25], as well as in this paper, we considered only the variant with D_3 symmetry, since it is the only one that provides a predictable and uniquely determined solution. The detailed procedure of obtaining such arrangements for frequencies 3<b<9, together with the tabular review of all the possible solutions are given. The research has shown that there are 30 ways to tile the triangle of 3<***b***<9 frequency by applying the maximum level of symmetry for the triangle, D_3 symmetry. The point is that, in this way, a wallpaper group ***p6m*** [9], [27] is formed, which coincides with the arrangement of the triangular faces LPT within the CC II's net. Thus, by placing the subdivided LPT back to the cupola's net, the section of a k-uniform tiling [14], [11], [25] is obtained. Folding the net into the 3D structure, CC II, provides a spatial tiling of the polyhedron's lateral surface. In this manner, we get a solution in which with the use of only two regular polygons, triangle and hexagon, different patterns and spatial solutions can be obtained. Serial production and prefabrication of just two shapes, makes such tiled structure easy and convenient for assembly.

What makes this study innovative compared to the previous research, is an attempt to find new, diverse solutions within the already existing tiling which consists of D_3 subdivided triangles, so that such solutions could suit even more successful architectural design. By color intervention, by using different materials, or by the omission of certain tiles from the existing layout, we can "paint" on the surface of these cupolae, and thus create patterns at our own will, without relying solely on the final result when assembling identically subdivided triangles.

5. RESULTS

Multilateral symmetry, even with just two planes of symmetry is often avoided in contemporary architectural solutions. The reasons are twofold: one concerns the possible confusion in the orientation for some users of such buildings, and the other concerns monotony in the visual experience. Since all the CC II are multilaterally symmetrical, and the subdivisional solutions for the ***p6m*** wallpaper group also have the highest level of symmetry, we propose in this paper that, within the same layout of the tiles, this symmetry can be changed or even distorted in order to get more interesting solutions.

Let us start from the existing patterns obtained by the method elaborated in paper [25]. Given that there are 14 representatives of CC II and 30 ways of tiling the LPT with the given frequencies 3<***b***<9, we have a total of 420 different solutions. They can be further modified by using different materials, colors, or by deliberate omission of some tiles, as aforementioned.

In the study [25], several examples are given as illustrations of the final appearance of the tiled CC II. The layout of the hexagonal tiles in these solutions is uniform and covers the entire surface of the cupola. Some of the solutions are suggested as architectural design of cupolas, domes, roofs, halls, or alike.

So, let us consider an example: augmented CC II-5M, subdivided by ***b***-frequency LPT with D_3 symmetry (as in Fig 4, framed cases). The example of such a tiling is given in Fig. 5.

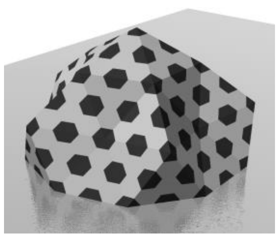

Fig. 5 An example of the augmented CC II-5 with 6-frequency triangular faces [25], with tri-hexagonal tiling.

This study aims to show that the monotony of self-repeating D_3 patterns on LPT can be broken by:

omitting a complete set of triangular tiles,

1) choosing the CC II which concave shape considerably vary, depending on the view point,
2) changing the color of different tiles.

1) Let us start from the case where D_3 symmetry is not disturbed, but the outline of the cupola gets new appearance and cupola itself takes on a porous structure.

When we remove the triangles from the observed tiling, we get new shapes in 3D assemblage (Fig. 6), provided that we choose physically sustainable solutions, i. e. those where hexagonal tiles touch by the edges. Thus, the hexagonal tiles (slabs, panels) are creating a continual, monolithic structure that can exist as a standalone composition, which may then serve as an exostructure [1], increasingly present in contemporary architecture [19]. They can also find application in façade design, shades design, solar design [17], or alike.

The light-shadow ratio in these cases adds a new layer to the perception of the shape and opens up a new room for testing optimal solutions in this direction.

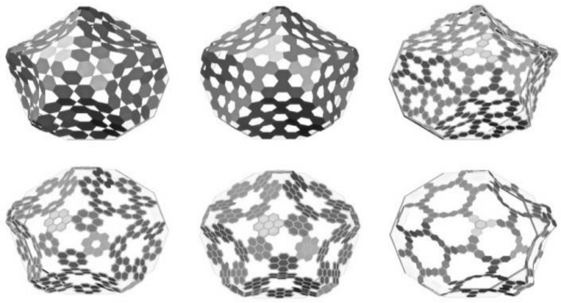

Fig. 6 Structures obtained by omitting the triangular tiles in the tiled deltahedral surface of augmented CC II-5M.

2) If we turn to the reasons for tiling CC II rather than some other deltahedral surfaces, we can see that, thanks to the concave shape of CC II, the appearance of the tiled cupola itself may vary significantly, depending on the viewing angle, i.e. the points from which we are observing it. The shapes of CC IIs with lesser number of sides in the basic polygon (CC II-4, CC II-5), make these differences more obvious. The alterations in what has been seen during the movement make monotony in observation of the tiled surface to be avoided. Also, with the same number of LPT triangles, and of consequently required tiles, we can make two different types of cupola: CC II-m and CC IIM (Fig. 7).

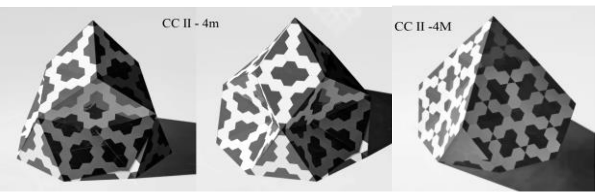

Fig. 7 CC II-4m tiled by 9-frequency LPT, seen from the different viewpoints, and CC II-4M with the same LPT.

3) From the former examples that use the entire set of identical tiles in the LPT subdivision, which are certainly most suitable for assembly, because just two types of tiles are employed: triangular and hexagonal, let us move to modified solutions. We modified them by changing color (material) or number of tiles, but not their general disposition. We took 2-uniform, trihexagonal tiling, or hexadeltille [5], [6], i.e. its 3D assemblage in augmented CC II-5M (Fig. 5 [25]), and changed its pattern by color.

We give just a few examples of how the uniformly subdivided cupolae can be transformed and redesigned.

In the first example, given in Fig. 8 a, we used hexagonal tiles in two colors. The ones of the same color are attached by the edges so that they form strips. By alternating these tiles, the stripes can be longitudinal or transverse.

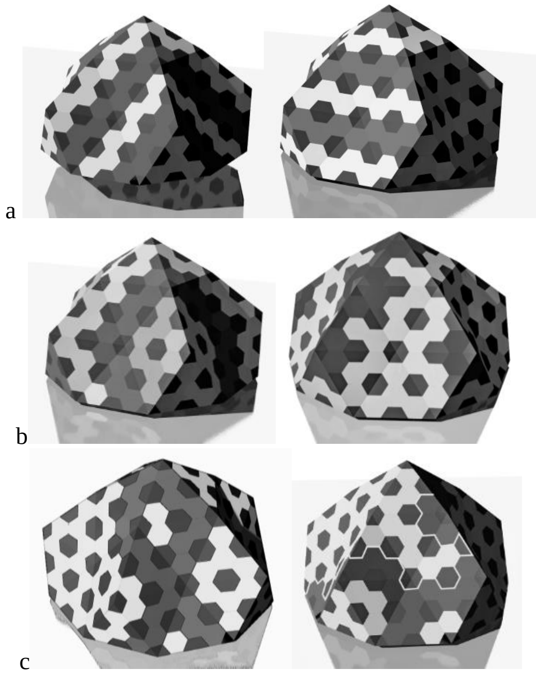

Fig. 8 Augmented CC II-5M, tiled by 6-frequency LPT and modified by color intervention.

The vertices of CC II can be highlighted in color, so that we can get a play of shades by color gradation, or forming floral patterns around the vertices (Fig. 8 b).

With a deliberate arrangement of differently colored tiles, we can form various figures, ornaments and decorations on the surface of CC II, symmetrical or asymmetric (Fig. 8 c).

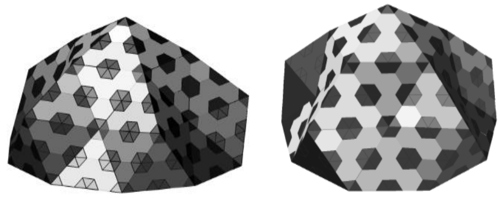

Fig. 9 Augmented CC II-5M, tiled by 6-frequency LPT, modified by using 4 shades of the same color of hexagonal tiles.

The color gradation can be performed throughout the surface of the cupola, or partly, in transverse or longitudinal direction, as shown in Fig. 9.

In Fig.10, we give a few more ideas to indicate how different patterns can be formed, using color interventions together with the omission of hexagonal tiles. Asymmetric solutions can be particularly interesting, because they offer nearly infinite possibilities for exploring visually satisfactory results.

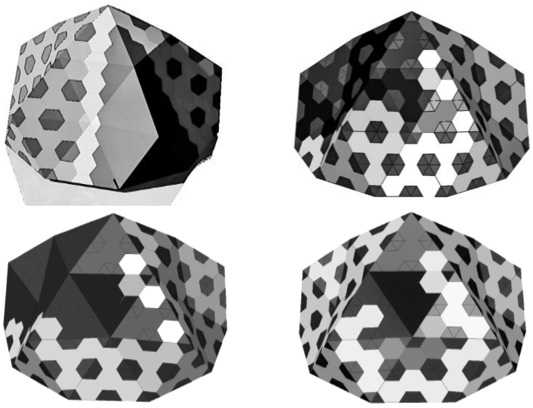

Fig. 10 Augmented CC II-5M, tiled by 6-frequency LPT modified by symmetric and asymmetric omitting of the hexagonal tiles.

Finally, in search of the solutions suitable for application in architecture, we can use just segments of the tiled surface of CC II, which will participate in the visual segment of the facade design, e.g. as decorative panels, brise-soleil, etc. (Fig. 11).

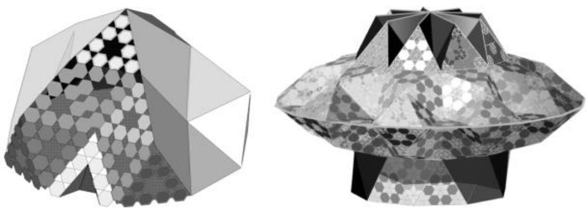

Fig. 11 Fragments of CC II tiling used as a façade design.

6. FURTHER RESEARCH

This research can be further developed in several directions:

- toward dealing with higher frequencies subdivisions;
- toward generating random permutations of tiles, including random orientations of the triangles in the lateral surface, rendering generative design of patterns;
- toward considering other symmetric, but not D_3 symmetric solutions with symmetry levels lower than D_3 (Fig. 12 a)
- toward substituting some of the triangular tiles by the same tiling as in the initial LPT, thus transferring the problem into the realm of fractals (Fig. 12 b);

Also, even greater freedom in combining color of the tiles can be explored within the subdivision of the higher frequency LPT, which represents a fragment of regular, hexagonal tiling (Fig. 12 c).

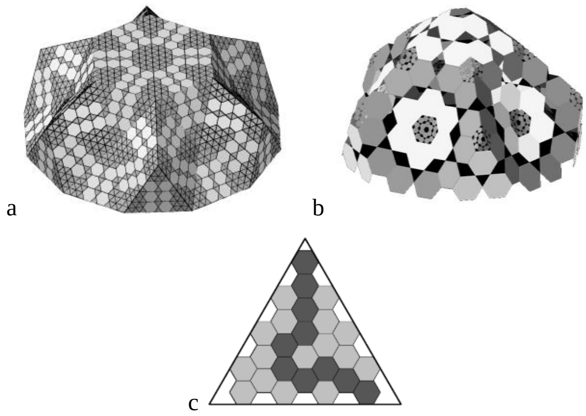

Fig. 12 a) An example of tiled surface of CC II-6M with reflexive symmetry b) Fractal subdivision of LPT on CC II-5M, c) LPT subdivided as a section of uniform, hexagonal tiling.

7. CONCLUSION

As an extension of the research begun with consideration of the most symmetric layout of tiles that can be applied to the triangular faces (LPT) of the CC II, which resulted in the segments of k-uniform tilings, this paper gives a broader view to the possibilities for their application in architecture. In order to get more diverse solutions which can break the monotony of uniform tiling, several interventions were considered. Omitting the triangular tiles from the surface, hexagonal ones can form an autonomous structure. With the alternation of different colors and materials, or with the omission of certain tiles, we can alter the symmetry of the tiling solution and redesign the appearance of the cupolae. The subject itself gives room for many new directions of research, from considering higher LPT frequencies, fractal tiling, or lower symmetry subdivisions of LPT, through generative design, to artistic "painting" of the lateral surface of the observed tiled cupola.

ACKNOWLEDGEMENT

The research is financially supported by Ministry of Education, Science and Technological Development of the Republic of Serbia, grant No III 44006.

REFERENCES

[1] Calero Daniel. 2014. *The Exostructure*. Symbiotic Cities International Design Ideas Competition. 30/09/2014. https://www.symbioticcities.net/index.cfm?id=65036&modex=blogid&modexval=18536&blogid=18536 Accessed 29 June 2018.

[2] Chavey, Darrah. P. 1984. *Periodic tilings and tilings by regular polygons* (Doctoral dissertation, University of Wisconsin, Madison).

[3] Chavey, Darrah, P. 1989. *Tilings by Regular Polygons -II: A Catalog of Tilings*, Computers & Mathematics with Applications. Vol. 17 No 1-2: 147–165.

[4] Clinton, Joseph, D. 1971. *Advanced structural geometry studies. Part 1: Polyhedral subdivision concepts for structural applications*. I:1-96.

[5] Conway, John. H. 1992. *The Orbifold Notation for Surface Groups: 438-447*. Cambridge University Press.

[6] Conway, John H, Heidi Burgiel, Chaim Goodman-Strauss. 2008. *The Symmetries of Things.* Wellesley, MA: A K Peters, Ltd.

[7] Coxeter, Harold Scott Macdonald. 1973. *Regular polytopes*. Courier Corporation.

[8] Critchlow, Keith. 1969. *Order in space: a design source book*. London: Thames and Hudson.

[9] Fedorov, Evgraf Stepanovich. 1891. *Симметрія на плоскости (Symmetry in the plane)*, Записки Императорского С.-Петербургского минералогического общества (Proceedings of the Imperial St. Petersburg Mineralogical Society), series **2, 28** : 345–390 (in Russian).

[10] Fuller, Buckminster R. 1982. *Synergetics: explorations in the geometry of thinking*. Estate of R. Buckminster Fuller.

[11] Galebach, Brian L. 2002. *N-uniform tilings*, Fri, 31 Dec 1999 23:59:59 GMT, Copyright © 2002-2018 by Brian L. Galebach. http://probabilitysports.com/tilings.html Accessed 29 June 2018.

[12] Ghyka, Matila Costiescu. 1946. *The geometry of art and life*. Courier Corporation.

[13] Goldberg, Michael. 1937. *A class of multi-symmetric polyhedra*. Tohoku Mathematical Journal, First Series, 43: 104-108.

[14] Grunbaum, Branko, Geoffrey C. Shephard. 1977. *Tilings by regular polygons*, Mathematics Magazine Vol. 50. No5: 227-247.

[15] Hamkins, Joel David. 2016. *There are no regular polygons in the hexagonal lattice, except triangles and hexagons* http://jdh.hamkins.org/no-regular-polygons-in-the-hexagonal-lattice/ Accessed 04. July 2018.

[16] Kepler, Johannes. 1619/ 1968. *Harmonices mundi*. Libri V. Culture et Civilisation.

[17] Kuang, Cliff. 2009. *Quick, hide the solar panels*! – Fast Company. https://www.fastcompany.com/1444721/quick-hide-solar-panels Accessed 04. July 2018.

[18] Misic, Slobodan, Marija Obradovic, Gordana Dukanovic. 2015. *Composite Concave Cupolae as Geometric and Architectural Forms*, Journal for Geometry and Graphics Vol.19. No 1: 79-91. Vienna: Heldermann Verlag.

[19] Morini, Lucio and *GGMPU Arquitectos*. 2012. The Bicentennial Civic Center (Cordoba, Argentina). ArchDaily. https://www.archdaily.com/290070/bicentennial-civic-center-lucio-morini-ggmpu-arquitectos Accessed 29. June 2018.

[20] Obradovic, Marija, Slobodan Misic. 2008. *Concave Regular Faced Cupolae of Second Sort*, Proceedings of 13th ICGG Dresden, Germany. El. Book: 1-10.

[21] Obradovic, Marija, Slobodan Misic, Maja Petrovic. 2012. *Investigating Composite Polyhedral forms obtained by combining concave cupolae of II sort with Archimedean Solids*, Proceedings of 3rd International Scientific Conference MoNGeometrija 2012:109 – 123.

[22] Obradovic, Marija, Slobodan Misic, Branislav Popkonstantinovic, Maja Petrovic, Branko Malesevic, Ratko Obradovic. 2013. *Investigation of Concave Cupolae Based Polyhedral Structures and Their Potential Application in Architecture, Technics Technologies Education Management* Vol. 8. No.3. 8/9: 1198-1214.

[23] Obradovic, Marija, Slobodan Misic, Branislav Popkonstantinovic, Maja Petrovic. 2011. *Possibilities of Deltahedral Concave Cupola Form Application in Architecture*. Proceedings of ICEGD Conference, Iasi 2011, (Buletinul Institutului Politehnic din Iasi, Publicat de Universitatea Tehnica „Gheorghe Asachi" din Iasi), Tomul LVII (LXI) Fasc. 3:123-140.

[24] Obradovic, Marija. 2006. Konstruktivno – *Geometrijska obrada toroidnih deltaedara sa pravilnom poligonalnom osnovom (Constructive-geometrical elaboration on toroidal deltahedra with regular polygonal bases).* Doctoral Dissertation, University of Belgrade, Faculty of Architecture.

[25] Obradovic Marija. 2019. Tiling the Lateral Surface of the Concave Cupolae of the Second Sort, Nexus Network Journal, Springer. First online: November 2018. https://link.springer.com/article/10.1007/s00004-018-0417-5

[26] Pickover, Clifford A. 2009. *The math book: from Pythagoras to the 57th dimension, 250 milestones in the history of mathematics*. New York: Sterling.

[27] Polya, George. 1924. *Uber die Analogie der Kristallsymmetrie in der Ebene. (On the analog of crystal symmetry in the plane),* Zeitschrift für Kristallographie, 60: 278–282.

[28] Wenninger, Magnus J. 1979. Spherical models. Vol. 3. Courier Corporation.

Author:

Associated Professor Marija Obradovic, professor, University of Belgrade, Faculty of Civil Engineering, Department of Mathematics, Physics and Descriptive Geometry, E-mail: marijao@grf.bg.ac.rs