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VARIATIONS OF CONCAVE PYRAMIDS OF SECOND SORT WITH AN EVEN NUMBER OF BASE SIDES

Abstract: *Concave pyramids of second sort (CP II) are formed similarly to concave cupolae of second sort (CC II) with a comparable logic of emergence to the regular-faced convex pyramids. They are formed by enclosing the space over a regular polygonal base by the lateral deltahedral wrapper consisting of equilateral triangles meeting in common apex. The plane net is composed of double row of equilateral triangles, so it is possible to form a regular faced pyramid even with bases greater than pentagonal, whereat the obtained polyhedron will be concave. In this paper we focus on a type of concave pyramids of second sort formed over the basic polygons with an even number of sides, type CP II-B which has half the number of lateral unit cells than the type CP II-A.*

Key words: *concave polyhedron, concave pyramid, deltahedral lateral surface, regular polygonal basis*

1. INTRODUCTION

In this paper we discuss the emergence, formation and description of polyhedral forms: concave pyramids of the second sort - type B (CP II -B), and forms derived from these solids by their duplications (through plane reflection and rotoreflection) and elongations.

Generally, concave pyramids of second sort (CP II) are polyhedra which follow the method of generating concave cupolae of second sort (CC II) [8], using the same method of folding the plane net which consists of duplicated series of equilateral triangles, as shown in **Fig. 1,** closed over a regular polygonal base. Hence, these polyhedra are regular-faced and have dihedral symmetry, because the line through the apexes intersects the base at its center. One of the basic conditions for the formation of solids is the continuity of the interior space, so we respected this criterion. The difference in the process of formation, compared to CC II**,** is that the unit element which forms the solid by its radial array, in this case is a spatial pentahedral cell instead of hexahedral one.

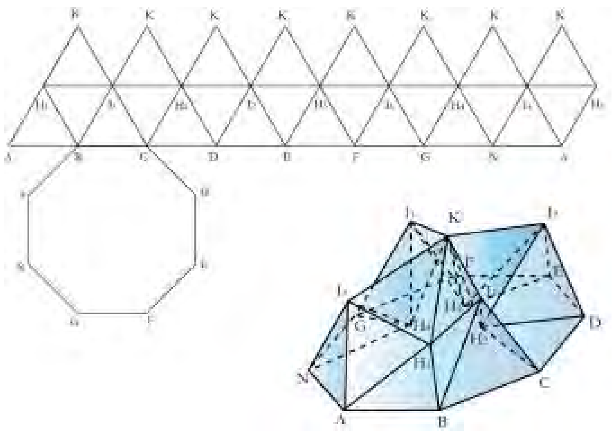


Fig. 1 the Formation of CP II-B by Folding the Plane Net of Equilateral Triangles

Among CP II we distinguish two types of the polyhedra: CP II-A [12] which arises by the radial arrangement of the unit cells over each side of the base polygon, and the type B, considered in this paper, generated by arrangement of the unit cells over every other side of the base polygon. Thus, this type of concave pyramids of second sort formed over the same basic polygon with an even number of sides will have half the number of lateral unit cells than the type CP II-A.

The structures with lateral surfaces corresponding to the polyhedra concerned in this paper and the method of their generation withal, are elaborated in detail in [11], wherein they are called "the core", as a part of more complex structures - toroidal deltahedra. The construction method for obtaining the vertices positions, the geometric basis for setting a numerical algorithm for all their parameters, and the visual display of these forms are also given in [11].

In this paper we consider the unique group of solids, CP II-B, their generation, properties, representatives and variations, with the intention to complete the group of concave polyhedra of the second sort, and to complement the group (class) of M-m polyhedra [6]. We approach the issue from an engineering point of view, as similar problems were treated in [1], [1], [3], [4] and [9].

2. THE GENERATION OF CP II-B

Concave pyramid of the second sort can be formed over a regular polygonal base, starting from n=6 to n=9. Unlike the CP II-A which can be formed over any of the given base (n=6, n=7, n=8, n=9) and for each of them they have two variations - one with greater height, CP II-AM and another with a lesser height, CP II-Am [12], CP II-B can be formed only over the bases with an even number of sides and have only one variant for each base, as will be explained hereinafter. The difference occurs because CP II-B is generated so that the observed unit pentahedral cell (Fig. 2) is distributed over every other side of the base polygon. Hence, this type of CP II can be completely formed (closed) only over the polygons with an even number of sides, n=6 and n=8. Although the segments of the lateral surfaces can be formed also over bases n=7 and n=9, they do not meet the requirement for the formation of the solid under the given conditions of dihedral symmetry. The lateral surface of the CP II-B can also be formed over the base n=10, but in this case the side faces intersect the base polygon, which excludes the possibility of closing a continuous interior space within the solid. Nonetheless, this structure can be used as part of a related group of concave pyramids, if elongated.

2.1. Constructive procedure

As given in the Fig 1, by folding and creasing the plane net consisting of n/2 pentahedral cells (i.e. five equilateral triangles arranged around the common vertex H) linked by equilateral triangles over an n-sided regular polygon, there can occur two positions of the observed unit pentahedral cell ABJKIH (marking is retained related to [11]) (**Fig. 2**).

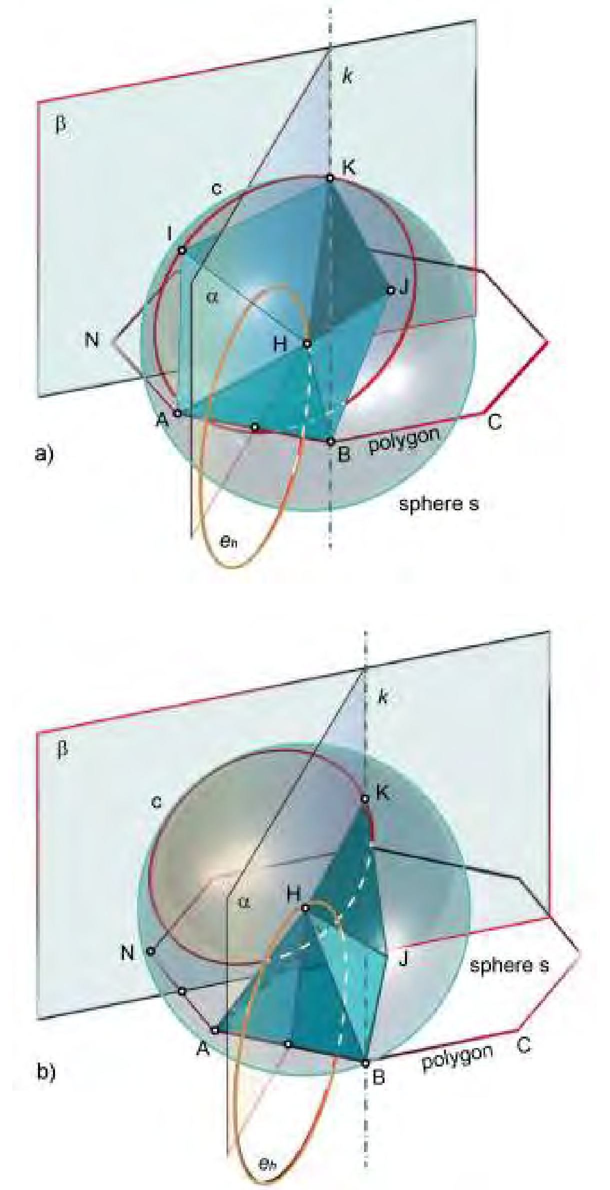


Fig. 2 The Origin of CP-II-B:
a) The Pentahedral Cell with the Indented Central Vertex H,
b) The Spatial Pentahedral Cell with the Protruding Central Vertex H

The one is obtained when the central vertex H of the unit pentahedral cell is indented into the interior of the solid (**Fig 2-a**), while the other occurs when the central vertex **H** is protruding to the exterior (**Fig 2-b**). In the case of CP II-B, these two positions of the unit cell alternate during the polar arrangement around the axis *k* of the solid. This means that, unlike CP II-A, where the same unit cells adjoin one after another and where there is a difference in the type and height of solid, depending on whether the vertex H is concave or convex, so there exists CP II-AM and CP II-Am, now there is no difference, because it will always appear one concave and one convex vertex of the identical cell in the same lateral surface. Since both positions of the cells converge in a common vertex K, the height of the CP II-BM and CP II-Bm will be identical, whether the vertex H is concave over the side AB, or over the adjacent side (AN, BC, etc.).

The essence of the constructive procedure described in [11], for determining the vertices' positions and all the linear and angular parameters needed for generation of CP II, lies in setting the spheres *s* of radii $R=a$ in the vertices of the spatial pentahedral cell ABIJKH.

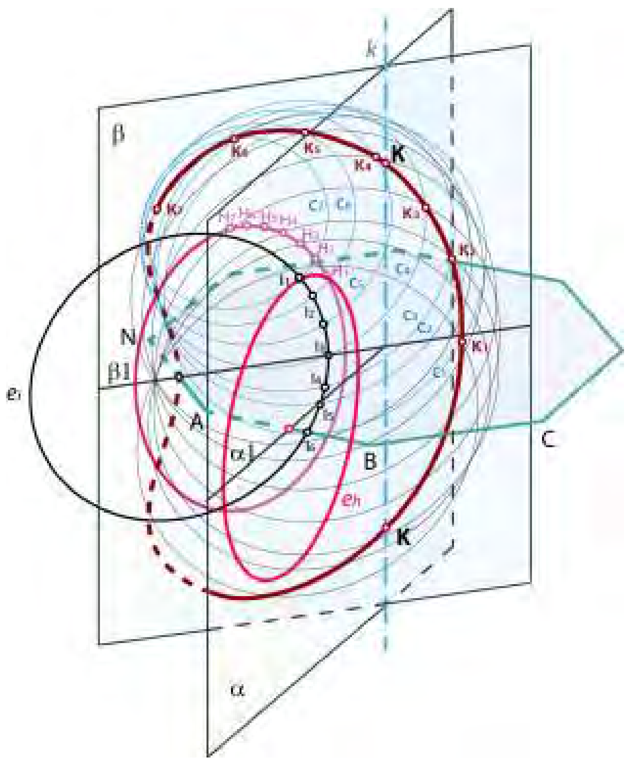


Fig. 3 The Construction of the Vertex K trajectory

Let us assume that the initial base polygon is set in the plane x-y (**Fig. 3**). The positions of its vertices A, B, C and so on, are known and their height is $h_{A,B}=0$. We need to find the height h_H of the vertex H, then $h_{I,J}$ for the vertices I and J, and h_K for the vertex K. As the vertex H lies in the intersection circle e_h of the spheres set in A and B, which is consequently the circle of equilateral triangle's vertex H rotation around the axis AB, the radius *r* of the circle e_h is: $r=a\sqrt{3}/2$. Accordingly, the vertex I will be found on the congruent circle e_i of its rotation around the axis AN. The position of the vertex K is found by setting new spheres (*s*) of $R=a$ in the vertices H and I, so that the intersection points of the circles *c* ($c=s_h \cap \beta$) and *i* ($i=s_i \cap \alpha$) obtained as plane sections of these spheres with the symmetry planes (β, α) of the adjacent sides of the base polygon, we find the position of the vertex K, depending on whether the vertex H is concave or convex.

Further construction relies on the iterative procedure based on setting up series of spheres *s(R=a)* centered on the circles e_i, in the assumed positions of the vertices I_n, and then the sequent series of the spheres *s(R=a)* set in the obtained positions of the vertices H_n on the circle e_h. The intersection points of the circles c_n and i_n, will give a trajectory of the vertex' K positions in the plane β (**Fig. 3**).

The obtained trajectory, unlike the cases of CC II and CP II-A, where it is an octic curve, now is a quartic curve – the degenerated octic curve on two overlapped quartic curves. This happens not only because of the symmetry of the structure, but also due to the alternation in the arrangement of pentahedral cells around the axis *k*. The intersection point of this curve and the axis *k* of the CP II-B gives the exact position of the vertex K.

2.2. Numerical algorithm

Based on the construction described in [11], a numerical algorithm is set, by which the iterations were performed in Microsoft Excel. Based on it, we obtained accurate positions of all vertices, angular and linear parameters of CP II-B. This algorithm has confirmed the geometric construction, also proving that the heights of both types of CP II-B are identical.

2.3. The outlook of the CP II-B

As already mentioned, it is possible to form only two representatives of CP II-B: CP II-6B of base n=6 and CP II-8B of base n=8. We can also form deltahedral lateral surface of base n=10, but not the whole CP II-10B.

In the **Fig. 4** are given three orthogonal projections of CP II-6B, as well as its 3D model.

The **Fig. 5** shows, as in the previous example, three orthogonal projections of CP II-8B and the 3D model of the same structure.

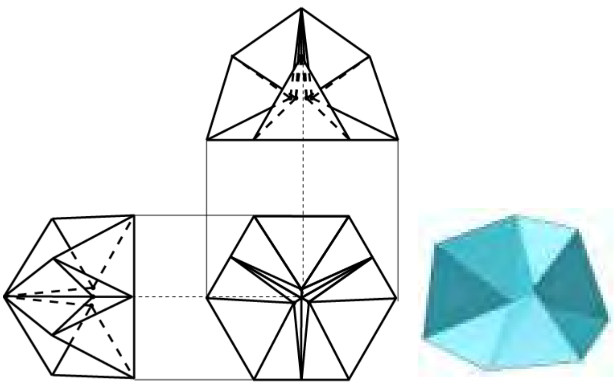


Fig. 4 Hexagonal Concave pyramid of the Second Sort: CP II-6B –Orthogonal Projections and 3D Model

The **Fig. 6** shows the orthogonal projections of just the lateral surface formed over regular polygon n=10, along with its 3D model, in order to present the structure which will participate in the composition of possible elongated solids.

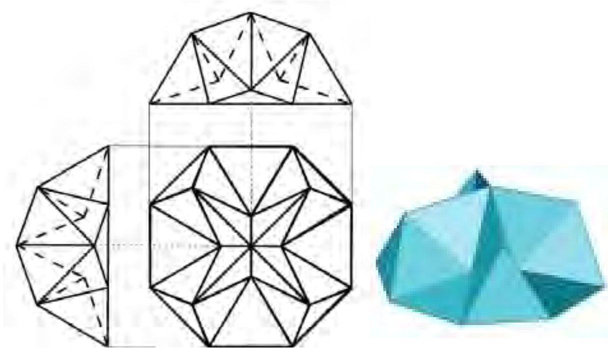


Fig. 5 Octagonal Concave pyramid of the Second Sort: CP II-8B –Orthogonal Projections and 3D Model

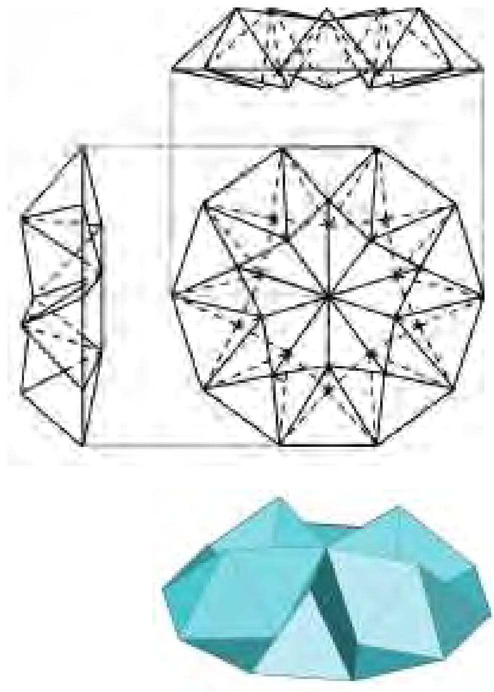


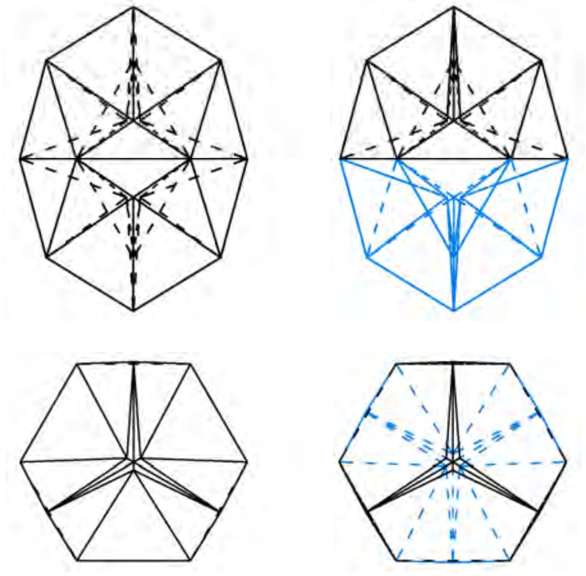
Fig. 6 Lateral surface of Decagonal Concave (by)pyramid of the Second Sort: Orthogonal Projections and 3D Model

3. THE VARITIONS OF CP II-B

CP II-B can be used as a component for the formation of composite polyhedral shapes: concave bipyramid of the second sort (CbP II–nB), elongated concave pyramids of the second sort (CeP II-nB), and elongated bipyramid of the second sort (CebP II-nB).

3.1. Concave bipyramids of the second sort

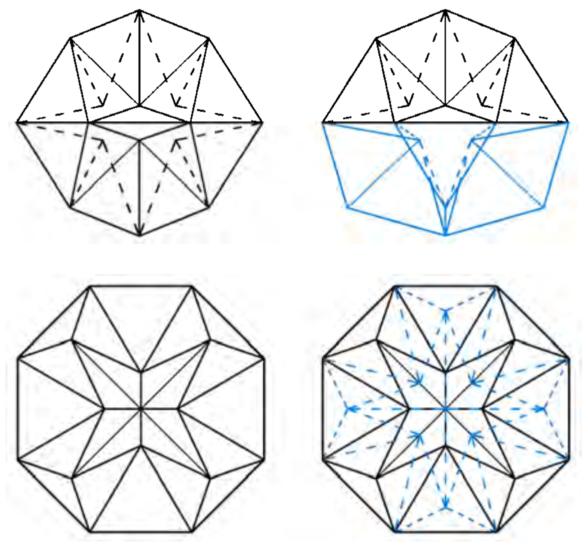
In order to create a face-transitive polyhedron, more precisely - deltahedron out of starting CP II-B, we can form bipyramids with the solids obtained by its plane or point reflections. In Johnson's classification of concave regular-faced solids [5]5], joining n-sided pyramid with its mirror-image base to base, we obtain orthobipyramid, while by joining it with its mirror-image, but of opposite orientation, we obtain gyrobipyramid.



a) b)

Fig. 7 Hexagonal Concave bipyramid of the Second Sort:
a) CobP II-6B, b) CgbP II-6B

In the same manner we obtain concave orthobipyramid of the second sort – type B (abbreviated: CobP-II-nB) and concave gyrobipyramid of the second sort – type B (abbreviated: CgbP-II-nB), as shown in **Fig. 7** and **Fig. 8.**



a) b)

Fig. 8 Octagonal Concave Bipyramid of Second Sort:
a) CobP II-8B, b) CgbP II-8B

3.2. Elongations

Using CP II-B as a basic unit, we can create diverse variations of concave polyhedra. By adding the appropriate polyhedral extensions, such as: prisms, antiprisms or concave antiprisms of second sort (CA II) [10], we can form four of the possible cases over just one of the bases (octagonal), as shown in **Fig. 9**:

- elongated concave pyramid of second sort (CeP-II-nB in **Fig. 9 a**),

- gyroelongated concave pyramid of the second sort (CgeP II-nB in **Fig. 9 b**) and

- conca-elongated [7][7] concave pyramid of the second sort, major (CcMeP-II-nB in **Fig. 9 c**), and

- conca-elongated concave pyramid of second sort, minorr (CcmeP-II-nB in **Fig. 9 d**).

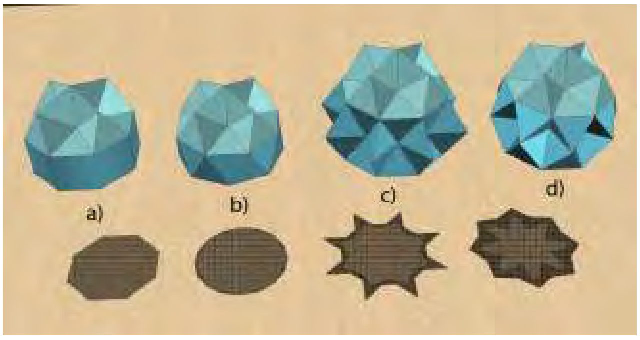


Fig. 9 Elongated Octagonal Concave Pyramid of Second Sort:
a) CeP II-8B, b) CgeP II-8B, c) CcMeP II-8B,
d) CcmeP II-B

By elongations of concave bipyramids, we obtain concave elongated, gyroelongated or conca-elongated pyramids of the second sort CebP II-nB, CgebP II-nB, CcMebP II-nB, CcmebP II-nB, respectively.

Thereby, in the cases of gyroelongated and conca-elongated concave bipyramids, we obtain deltahedral forms. In **Fig. 10** we present eight representatives of possible variations of the octagonal concave bipyramid of second sort, type B (CbP II-8B). It should be noted that the cases of b) and f) are not identical but chiral; i.e. they can not be lead to congruence by rotation, but only by plane reflection.

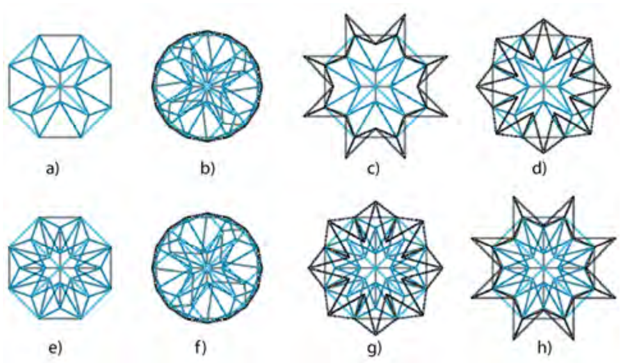


Fig. 10 Elongated Octagonal Concave Bipyramids of the Second Sort, type B:
a) CeobP II-8B, b) CgeobP II - 8B, c) CceMobP II - 8B
d) CcemobP II – 8B, e) CegbP II – nB, f) CgegbP II – nB,
g) CceMgbP II – nB, h) CcemgbP II – nB

In Fig. 11 we show the rendered 3D models of elongated octagonal concave orthobipyramids, and in **Fig. 12** we show 3D models of elongated octagonal concave gyrobipyramids.

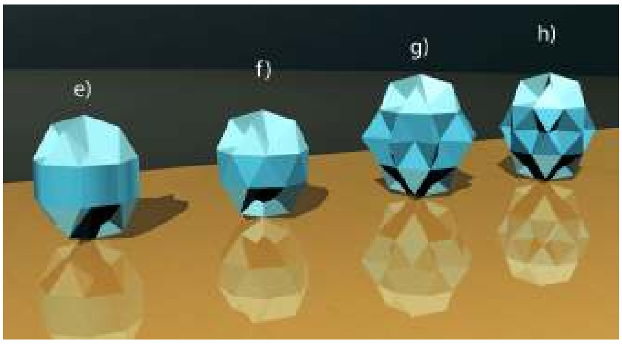


Fig.11. Elongated CbP II-8B (a, b, c, and d from Fig. 10)

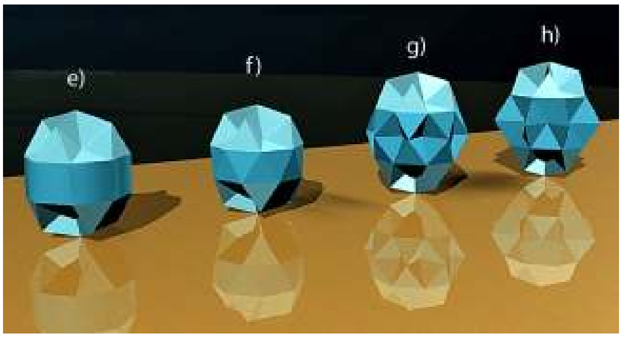


Fig. 12. Elongated CgbP II-8B (e, f, g and h from Fig. 10)

In the **Table 1** we present the overview of the possible polyhedral shapes formed using the units of CP II-B. We find 42 new solids which respect the preset criteria for the formation of the concave polyhedral shapes obtained in the manner described in the introductory chapter.

Table 1

The group of polyhedral forms obtained using CP II-B

Type		Marks of the cases	n 6	n 8	n 10
Concave Pyramids Of the second Sort and their elongations	1	CP II - nB	✓	✓	-
	2	CeP II - nB	✓	✓	✓
	3	CgeP II - nB	✓	✓	✓
	4	CceMP II - nB	✓	✓	✓
	5	CcemP II - nB	✓	✓	✓
Concave Bipyramids Of the Second Sort	6	CobP II - nB	✓	✓	-
	7	CgbP II - nB	✓	✓	-
Elongated, Gyro-elongated And Conca-elongated Bipyramids Of the Second Sort	8	CeobP II - nB	✓	✓	✓
	9	CgeobP II - nB	✓	✓	✓
	10	CceMobP II - nB	✓	✓	✓
	11	CcemobP II - nB	✓	✓	✓
	12	CegbP II - nB	✓	✓	✓
	13	CgegbP II - nB	✓	✓	✓
	14	CceMgbP II - nB	✓	✓	✓
	15	CcemgbP II - nB	✓	✓	✓
Total number		42			

In this way, we complement and complete the group of convex polyhedra of the second sort. Furthermore, by defining additional polyhedral shapes, we supplement the family of M-m polyhedra that can, due to their compatibility, participate in further composition of complex polyhedral structures.

CONCLUSIONS

Using the method similar to one for the generation of CC II and CP II-A it is possible to obtain a type of concave pyramids of second sort (CP II-B), who have half the number of the pentahedral unit cells in their lateral surface, and can be formed only over the even-sided polygonal bases. There are two representatives of CP II-B, CP II-6B and CP II-8B, by whose variations it is possible to provide another 40 new concave polyhedra based on their geometry. Due to unification of their building blocks, these polyhedra may be suitable for further consideration in terms of feasible forms for use in architectural practice.

ACKNOWLEDGEMENT: Research is supported by the Ministry of Science and Education of the Republic of Serbia, Grant No. III 44006.

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