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MODELLING WITH GENERALIZED CYLINDER SURFACES

Abstract: *This paper presents several of the most interesting surfaces generated by the spine curve of a generalized cylinder surface, with known parameters exhibiting advantages related to the calculation and drawing of the surface.*

Key words: *generalized cylinder surface, hypocycloid, epicycloid, modelling surfaces, curve*

1. INTRODUCTION

The wish to erect buildings with a unique and spectacular architecture is supported by the diversification of building materials or the improvement of classical materials and the development of modern technologies as well. One of the issues always present remains the covering of wide surfaces.

2. MODELLING WITH GENERALIZED CYLINDER SURFACES

As it is shown in [2] and [4] a generalized cylinder surface has the following vector equation:

$$\Gamma(u,v) = C(u) + s_1(u)\varphi(v)n(u) + s_2(u)\psi(v)b(u) \quad (1)$$
$$u \in [a,b],\ v \in [c,d]$$

where,

$C(u) = (x(u), y(u), v(u))^T$, $u \in [a,b]$ is a regular curve, named the spine (guide) curve of surface $\Gamma(u,v)$ and

$$\gamma(v) = (\varphi(v), \psi(v))^T,\ v \in [c,d], \gamma(c) = \gamma(d) \quad (2)$$

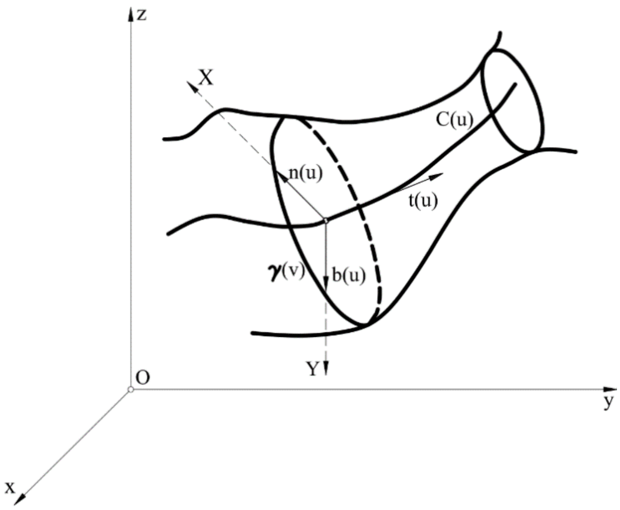

Fig. 1.

The curve $\gamma(v)$ is referred to the local coordinate system X, Y (see figure 1); the X, Y axes are on the directions of unit vectors $n(u)$ and $b(u)$ which are situated on the principal normal and binormal, respectively, on curve $C(u)$. These vectors, together with the unit vector $t(u)$ determine the Frenet trihedron of curve $C(u)$ and are given by the formulas:

$$t(u) = \frac{C'(u)}{|C'(u)|},\ b(u) = \frac{C'(u) \times C''(u)}{|C'(u) \times C''(u)|},\ n(u) = b(u) \times t(u) \quad (3)$$

The continuous and nonnegative functions $s_1(u)$ and $s_2(u)$, $u \in [a,b]$, scale the generating curve $\gamma(v)$ in the directions of vectors $n(u)$ and $b(u)$, respectively, and thus they have an important contribution to the shape of the generalized cylinder surface $\Gamma(u,v)$. One easily observes that the coordinate curve $\Gamma(u,v_0)$, v_0=constant, $v_0 \in [c,d]$mimics the graphic of function $s_1(u)\varphi(v_0) + s_2(u)\psi(v_0), u \in [a,b]$.

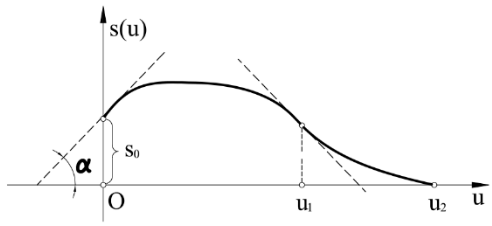

Fig. 2.

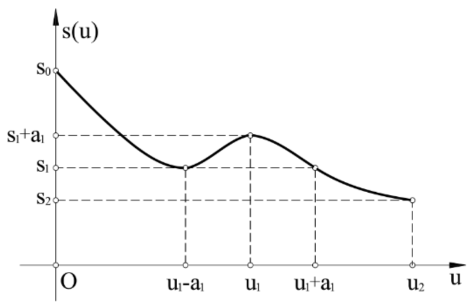

Fig. 3.

In what follows, we consider the special case $C(u)$ is the axis O_z, which rotates around itself with the angular velocity ω. Let us denote by α the angle of rotation of axis O_z, $\omega = \dfrac{\alpha}{b-a}$. In this particular case, we take

$n(u) = (\cos \omega u, -\sin \omega u, 0)^T$ and $b(u) = (\sin \omega u, \cos \omega u, 0)^T$ (4)

With the generating curve $\gamma(v)$, we have two cases:

1. Firstly we take $\gamma(v)$ an hypocycloid. The hypocycloid is the locus traced by a fixed point on a circle with radius r, that rolls, without slipping, on the interior of a fixed circle of radius R; $r = \dfrac{R}{n}$; n is a nonnegative integer, $n \geq 2$ and is represented by the equation:

$$\gamma(v) = r\begin{pmatrix}(n-1)\cos v + \cos(n-1)v, \\ (n-1)\sin v - \sin(n-1)v\end{pmatrix}^T \tag{5}$$

$v \in [0, 2\pi]$.

The surface $\Gamma(u,v)$ corresponding to these data has the following vector equation:

$$\begin{aligned}&\Gamma_h(u,v) = (0,0,u)^T + \\ &+ s_1(u) r((n-1)\cos v + \cos(n-1)v)(\cos \omega u, -\sin \omega u, 0)^T + \\ &+ s_2(u) r((n-1)\sin v - \sin(n-1)v)(\sin \omega u, \cos \omega u, 0)^T \\ &\quad u \in [a,b],\ v \in [0,2\pi].\end{aligned} \tag{6}$$

2. Next, we use the generating curve $\gamma(v)$, which is defined as locus traced by a fixed point on a circle with radius r, that rolls, without slipping, on the exterior of a fixed circle of radius R. Taking $r = \dfrac{R}{n}$, n being a nonnegative integer number, this curve has the vector equation:

$$\gamma(v) = \begin{pmatrix} r((n+1)\cos v + \cos(n+1)v), \\ r((n+1)\sin v - \sin(n+1)v)\end{pmatrix}^T \tag{7}$$

$v \in [0, 2\pi]$.

The generalized cylinder surface $\Gamma(u,v)$, generated by the curve (7) is represented by the vector equation:

$$\begin{aligned}&\Gamma_e(u,v) = (0,0,u)^T + \\ &+ s_1(u) r((n+1)\cos v - \cos(n+1)v)(\cos \omega u, -\sin \omega u, 0)^T + \\ &+ s_2(u) r((n+1)\sin v - \sin(n+1)v)(\sin \omega u, \cos \omega u, 0)^T \\ &\quad u \in [a,b],\ v \in [0,2\pi].\end{aligned} \tag{8}$$

In what follows, we need the length of curve $\gamma(v)$, $v \in [0,2\pi]$, and the area of the domain bounded by $\gamma(v)$, A, when $\gamma(v)$ is the hypocycloid and the epicycloid given by the vector equations (5) and (7) respectively.

In both cases one uses the formulas:

$$L = \int_0^{2\pi} \sqrt{\varphi'^2(v) + \psi'^2(v)}\,dv = n\int_0^{\frac{2\pi}{n}} \sqrt{\varphi'^2(v) + \psi'^2(v)}\,dv \tag{9}$$

and

$$\begin{aligned}A &= \frac{1}{2}\int_0^{2\pi} |\varphi(v)\psi'(v) - \varphi'(v)\psi(v)|\,dv = \\ &= n\int_0^{\frac{2\pi}{n}} |\varphi(v)\psi'(v) - \varphi'(v)\psi(v)|\,dv\end{aligned} \tag{10}$$

By direct calculus for the hypocycloid (5), one obtains:

$$L_h = \frac{8(n-1)R}{n} \tag{11}$$

and

$$A_h = \frac{(n-1)(n-2)\pi R^2}{n^2} \tag{12}$$

respectively, where $R = nr$.

For the epicycloid (7) it results

$$L_e = \frac{8(n-1)R}{n} \tag{13}$$

and

$$A_e = \frac{(n-1)(n-2)\pi R^2}{n^2} \tag{14}$$

respectively, where $R = nr$.

Also, we need the length $L(u)$ of an arbitrary cross-section of $\Gamma(u,v)$, $\gamma(u,v)$, u-parameter and the area of the domain bounded by the cross-section, denoted by $A(u)$.

For $L(u)$, we consider only the particular case, when $s_2(u) = s_1(u) = s(u) \geq 0$, $u \in [a,b]$. Using formula (9), $L = \int_0^{2\pi} \sqrt{s^2(u)\varphi'^2(v) + s^2(u)\psi'^2(v)}\,dv$, taking into account (11) and (13), results:

$$L_h(u) = \frac{8(n-1)R}{n} s(u),\ u \in [0, u_2] \tag{15}$$

and

$$L_e(u) = \frac{8(n+1)R}{n} s(u),\ u \in [a_1, b_1] \tag{16}$$

Respectively, the area $A(u)$ of any cross-section, considering (12) and (14) and for arbitrary $s_1(u) \geq s_2(u) \geq 0$, formula (10) gives:

$$A_h(u) = \frac{(n-1)(n-2)\pi R^2}{n^2} s_1(u) s_2(u),\ u \in [0, u_2] \tag{17}$$

and

$$A_e(u)=\frac{(n+1)(n+2)\pi R^2}{n^2}s_1(u)s_2(u),\ u\in[a_1,b_1] \tag{18}$$

respectively.

Finally, using formulas (20) and (26) from the paper [1], in the above hypothesis, the areas of the generalized cylinder surfaces (6) and (8), respectively, are given by the following formulas:

$$S(\Gamma_h)=\frac{8(n-1)R}{n}\int_a^b s(u)du \tag{19}$$

and

$$S(\Gamma_e)=\frac{8(n+1)R}{n}\int_a^b s(u)du \tag{20}$$

respectively.

Volumes of the solids bounded by the generalized cylinder surfaces $\Gamma_h(u,v)$ and $\Gamma_e(u,v)$, respectively and the planes $u=a$ and $u=b$, the formula (26) from [2] is given by:

$$V_h=\frac{(n-1)(n-2)\pi R^2}{n^2}\int_a^b s_1(u)s_2(u)du \tag{21}$$

$$V_e=\frac{(n+1)(n+2)\pi R^2}{n^2}\int_a^b s_1(u)s_2(u)du \tag{22}$$

Finally, we present four particular generalized cylinder surfaces of vector equations (6) and (8) corresponding to the following scale functions:

$$\text{I.}\quad \begin{array}{l} s_1(u)=s_2(u)=s(u;m;u_1;u_2;s_0) \\ u\in[0,u_2];\ m=tg\alpha \\ s(u;m;u_1;u_2;s_0)=\dfrac{mu_2+s_0}{u_2^2(3u_1-u_2)}u^2(u-3u_1)+mu+s_0; \\ u\in[0,u_2] \end{array} \tag{23}$$

$$\text{II.}\quad \begin{array}{l} s_1(u)=s_2(u)=s(u;a_1;s_0;s_1;s_2;u_1;u_2)= \\ =\begin{cases} s_1+\dfrac{s_0-s_1}{(u_1-a_1)^2}(u-u_1-a_1)^2;\ u\in[0,u_1-a_1] \\ s_1+\sqrt{a_1^2-(u-u_1)^2};\ u\in[u_1-a_1,u_1+a_1] \\ s_1+\dfrac{s_2-s_1}{(u_2-u_1-a_1)^2}(u-u_1-a_1)^2;\ u\in[u_1+a_1,u_2] \end{cases} \end{array} \tag{24}$$

In Figures 4 and 5, respectively 6 and 7, there are presented surfaces for whose generation a hypocycloid and an epicycloids and the S type I curve were used. Figures 5 and 7 present the same surfaces, seen from below.

The values of the parameters used to generate the surfaces in Figures 4 to 7 are: R=4, a=0, b=3, m=1, s_0=4, u_1=2, n=8.

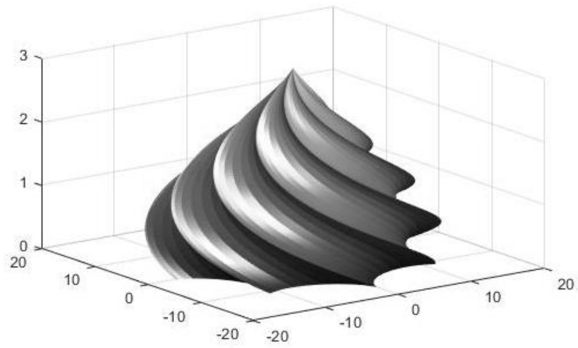

Fig. 4 Surface generated by the hypocycloid function of type I scale.

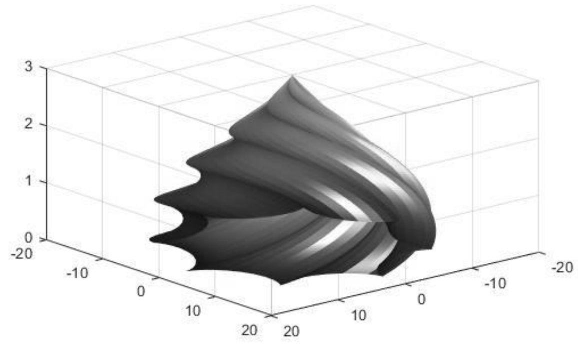

Fig. 5 Surface generated by the hypocycloid function of type I scale, seen from below.

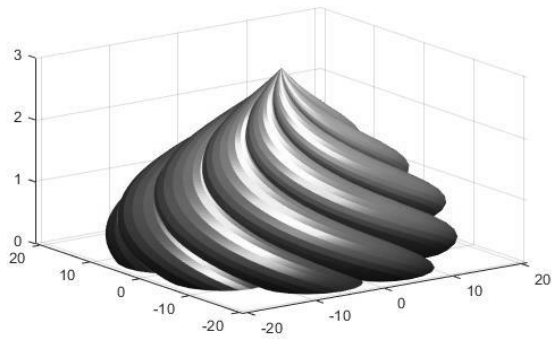

Fig. 6 Surface generated by the epicycloid function of type I scale.

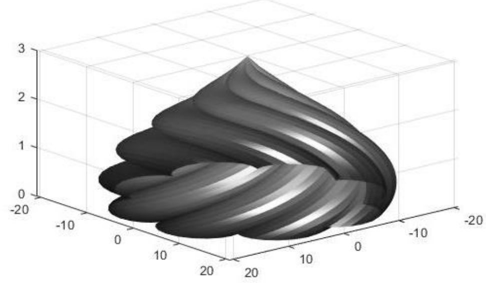

Fig. 7 Surface generated by the epicycloid function of type II scale, seen from below.

In Figures 8 and 9, respectively 10 and 11 there are given surfaces for whose generated a hypocycloid and an epicycloids were used, function of type II scale.

In Figures 9 and 11the same surface from Figures 8 and 10, but seen from below.

The values for the parameters used to generate the surfaces in Figures 8-11 are: R=4, a=0, b=4, m=1, s_0=2, s_1=1, s_2=0, u_1=2, a_0=0,25, n=6.

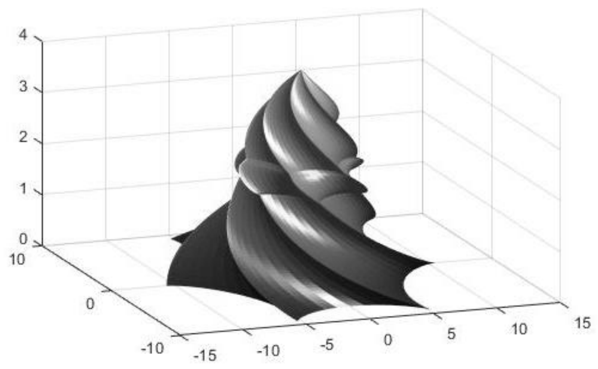

Fig. 8 Surface generated by the hypocycloid function of type II Curve.

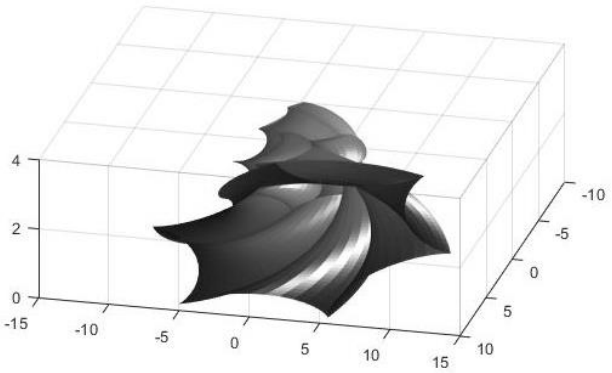

Fig. 9 Surface generated by the hypocycloid and type II Curves, seen from below.

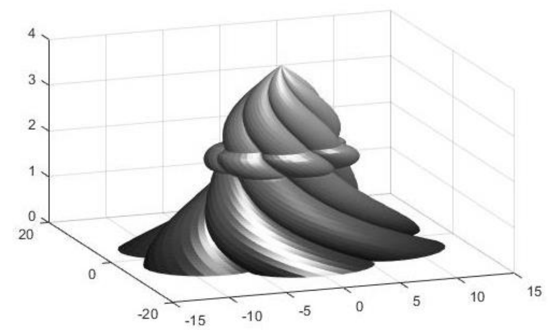

Fig. 10 Surface generated by the epicycloid and type II Curves.

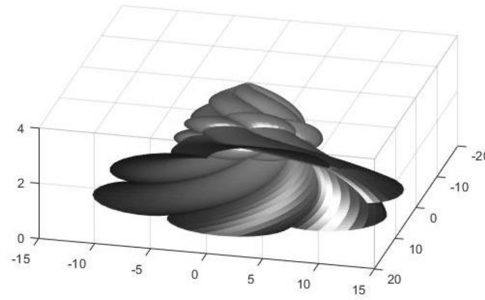

Fig. 11 Surface generated by the epicycloid and type II Curves, seen from below.

3. CONCLUSION

The analytical calculus of these types of coverings – with surfaces generated by algebraic curves with known parameters – can be made rigorously, taking into account that in any point on the surface one can write the equation of the section curve, of the tangent and normal to the former.

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