*Liliana LUCA*

# A STUDY OF APPROXIMATE DEVELOPMENT OF SPHERICAL SURFACES

***Abstract:*** *An interesting problem in descriptive geometry is the unfolding of the non-developable surfaces of revolution. In this paper we start from the two classic methods of approximate development of a sphere: the gore method and the zone method. For each of the two methods, we study the errors of approximation. It is considered an approximate development of sphere using gore method, in 3 variants, using successively: 4 meridian (vertical) planes (resulting 8 gores), 5 meridian planes (10 gores) and 6 meridian planes (12 gores). Also for the same sphere, the approximate development is obtained using successively: 7 level planes (resulting 8 zones), 9 level planes (10 zones) and 11 level planes (12 zones). For every variant of development, the error is calculated by comparing the area of approximate development and the theoretical area of the sphere. A computerized method based on Maple procedures is used. In the paper there is also a comparison between the errors of the two methods (gore method and zone method)..*

***Key words:*** *sphere, development, computerized method, errors.*

## 1. INTRODUCTION

Within the industrial field in order to manufacture technical pieces it is necessary to establish the theoretical semi-manufactured within the conditions of a bigger and bigger material economy, both for the foil pieces and for the laminated profile or pipe pieces. The pieces manufactured from foils are very often used within the industrial field and in these cases it is necessary to establish the whole area of the geometrical object. Some pieces have geometrical forms made up of rotation surfaces. In these cases there occurs the problem of establishing the whole area by approximate methods. In many cases it is necessary that the whole area be established very precisely and then we must choose the most accurate calculation method, the smaller and smaller error method.

We also carried out studies concerning the whole area of some rotation surfaces with esthetic form. The paper [7] proposes an approximate graphical method for determining the development of the rotational surface obtained by means of the deltoid. Studies about the development of rotation surfaces generated by astroid and hypocycloid with 4 branches, are presented in the papers [8] and [9]. Interesting studies on the generation of some esthetic rotation surfaces are given in papers [10] and [11] and in the future we also could study more on their application within the industrial engineering including the whole development problem.

The paper [4] proposes practical graphic methods provided by Descriptive Geometry to the development for two revolution surfaces: the revolution surface having an oval as median section and the revolution surface having an ordinary curve as median section. In 2015 we carried out a complex paper on the development theme, [6], containing more whole accurate and approximate area methods for the development of the different pieces used within the industrial engineering.

Surfaces and aesthetic curves have become interesting for many industrial applications. The paper [13] presents research on some geometric figures with aesthetic shapes. In recent years are used very much the computer assisted methods. An interesting problem of descriptive geometry is solved by two methods, method of the descriptive geometry and method with program Solid Works, [5]. The sphere development can be approximately developed by any one of the following methods: gore method, zone method, triangulation method. The gore method and the zone method is presented in many specialized papers, we mention for example papers [12] and [14].

## 2. GRAPHICAL METHODS FOR THE SPHERE DEVELOPMENT

### 2.1. Sphere development by the graphical - Gore method

The surface is divided into a number of equal vertical sections. Each section is considered as an arched segment. For the sphere shown in Figure1 it is presented the development determined graphically based on the gore method. When tracing the evolute it is used a number of auxilliary planes (level and vertical planes). It is presented in detail the construction for Gore- method.

P1, P2, ... P6 are the vertical planes. N1, N2, N3 are the level planes. For the geometric construction of a gore we have the following steps:

- there are built the auxiliary planes,
- it is drawn a line segment equal to $\pi$R (R is the radius of the sphere),
- It is divided into 8 equal parts,
- there are drawn perpendicular in the obtained points,
- there are measured on these perpendicular lines, equal segments with the bisecants corresponding to the appropriate circle arcs determined between two vertical level plane (P1 and P2),
- there are joined the points obtained with curved line and results a gore.

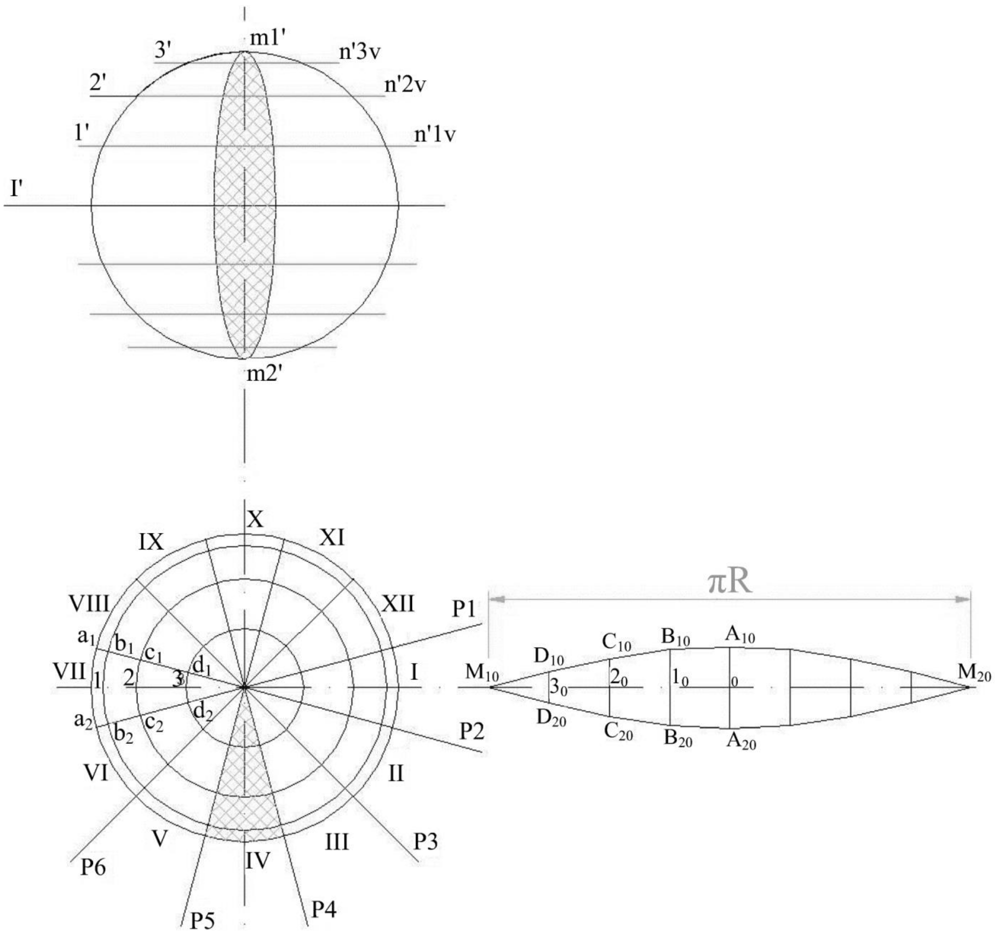


**Fig. 1** Sphere development. Gore method

### 2.2. Sphere development by the graphical - Zone method

The surface is divided into horizontal zones and each zone is developed as a frustum of a cone. The sphere represented in Figure 2 is decomposed into several zones using N1, N2, N3, N4 planes. The evolutes of the two calottes are replaced by the evolutes of some rotation cones and the evolute of the intermediate zones through the evolute of frustums of cone. The vertices S1 ', S2' and S3' of frustums of cone are at the intersection of the vertical axis of the sphere with the generatrices passed through the points 1' 2 ', 2'3' and 3'4 '. Generatrices S1'1 ', S2'2', S3'3 'and P1'4' serve as radius for the frustums of cone and the cone vertex.

## 3. APPROXIMATE DEVELOPMENT OF THE SPHERE USING COMPUTERIZED METHODS. ERROR ESTIMATION

It is considered a sphere approximate development using gore method, in 3 variants, using successively: 4 meridian (vertical) planes (resulting 8 gores), 5 meridian planes (resulting 10 gores) and 6 meridian planes (resulting 12 gores). For every variant of development, the error is calculated by comparing the area of approximate development and the theoretical area of the sphere. A computerized method based on Maple procedures is used.

The Maple procedures defined in [1], [2] and [3] apply to a surface generated around the z-axis (vertical axis). It is assumed that the curve is specified parametrically by

$$\begin{cases} x=x(t) \\ z=z(t) \qquad t\in[a, b] \end{cases} \tag{1}$$

and that $x(t) \geq 0$ for all $t\in[a, b]$. Consequently, the standard parameterization of this surface of revolution is given by equations (2):

$$\begin{cases} x = x(t)\cos(\theta) \\ y = x(t)\sin(\theta), \qquad t\in[a, b],\ \theta\in[\theta_1,\ \theta_2] \\ z = z(t) \end{cases} \tag{2}$$

(in the case of a complete rotation $\theta_1=0$ and $\theta_2=2\pi$).

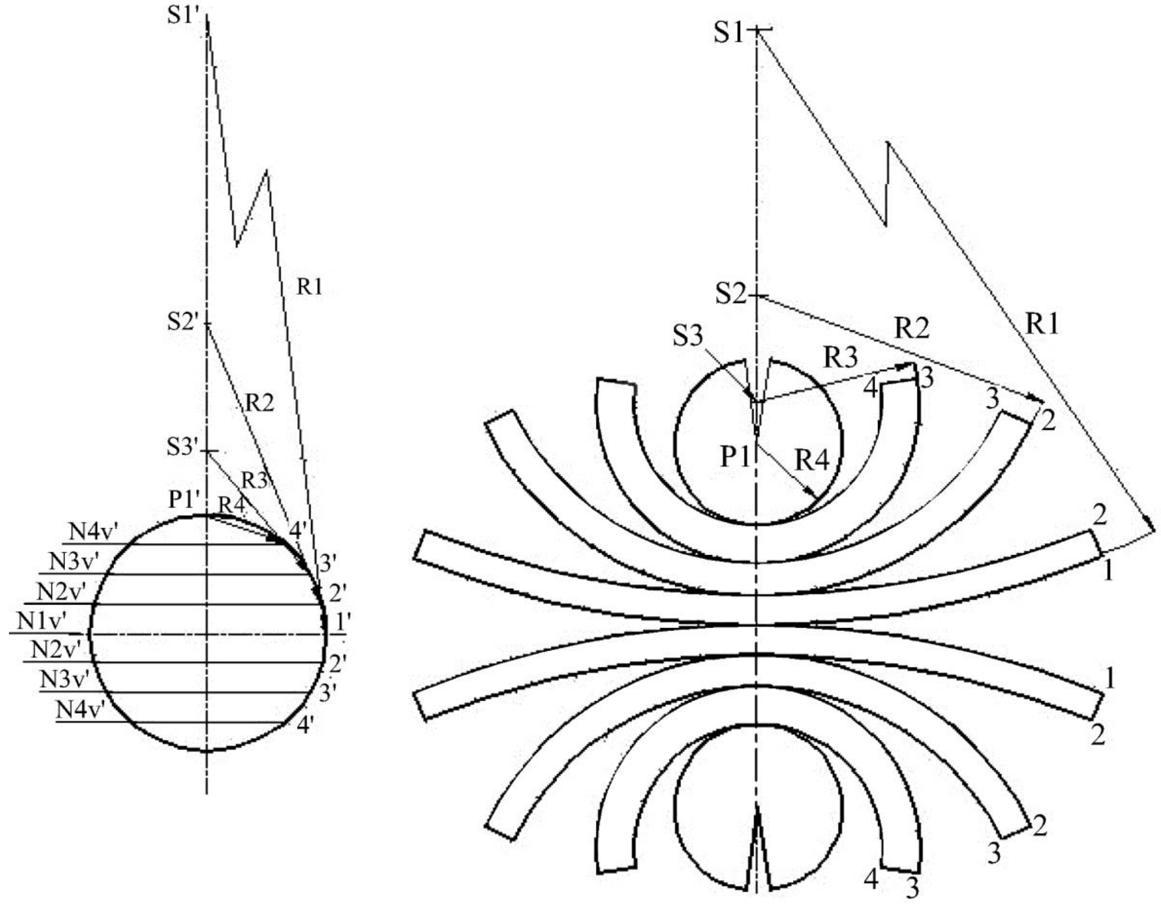


**Fig. 2** Sphere development. Zone method

In order to apply those procedures to a sphere of radius 100, we define in Maple:

```
> a:=-Pi/2: b:=Pi/2:
> theta1:=0: theta2:=2*Pi:
> r:=100:
> xt:=proc(t) RETURN(r*cos(t)) end
 proc:
> zt:=proc(t) RETURN(r*sin(t)) end
 proc:
```

The gore (or more generally, the arch shaped sector) is drawn joining the points obtained similarly to $A_{10}$, $B_{10}$, $C_{10}$, … (and symmetrically, $A_{20}$, $B_{20}$, $C_{20}$, …) in Figure 1 by a smooth curve. This curve joining the points $A_{10}$, $B_{10}$, $C_{10}$, … can be seen as a smooth function f with the property that $f(0_0)=A_{10}$, $f(1_0)=B_{10}$, $f(2_0)=C_{10}$, … The procedure approximation_gore [1] returns the approximation of f using spline functions (of degree d).

For estimate the error in the case of sphere approximate development using gore method, we use the following Maple procedures (that can be applied to any surface of revolution (2)):

```
> area_approximation_gore:=proc(x,z,
 parameter,theta1, theta2,n,d)
local len,m;
m:=nops(parameter);
len:=eval(curve_length(x,z,parameter[1
],parameter[m]));
RETURN(2*evalf(Int(approximation_gore(
x,z,parameter,theta1,theta2,n,d,X),X=0
..len)))
end proc;

> area_surf_rev:=proc(x,z,a,b,theta1,
 theta2)
local area;
area:=simplify(abs(theta2-theta1)*
evalf(Int(abs(x(t))*(diff(x(t),t)^2+di
ff(z(t),t)^2)^(1/2),t=a..b)));
RETURN(area)
end proc;

> error_approximation_gore:=proc(x,z,
 parameter,theta1,theta2,n,d)
local e, area, area_aprox;

area_aprox:=evalf(n*
area_approximation_gore(x,z,parameter,
 theta1,theta2,n,d));
print(`Area of the surface development
 approximation`, area_aprox);
area:=area_surf_rev(x,z,a,b,theta1,
 theta2);
```

```
print(`Surface of revolution area`,
 area);
print(`Absolute Error`,evalf(area-area
 _aprox));
print(`Relative Error (%)`, evalf(abs
 (area-area_aprox)/area*100))
end proc;
```

where x and z define the curve rotating around the z-axis, parameter is a list $[t_1, t_2,\ldots,t_{m+1}]$ that stores a division $(a=t_1<t_2<\ldots<t_{m+1}=b)$ of the interval [a,b], n is the number of equal sections obtained by vertical planes and d is the degree of spline functions used for approximate the gore. The procedure curve_length was defined in [1] and returns the length of the curve specified parametrically by (1).

For estimate the error in the case of sphere approximate development using gore method , we use an equidistant division of the interval [a,b] (obtained applying the procedure param_equid [1]) and the procedure error_approximation_gore in two cases: the gore is approximated by spline functions of degree 1 (the points are joined by lines), respectively by spline functions of degree 3.

In the case of an approximate development of sphere using gore method with 4 meridian (vertical) planes (resulting 8 gores), Figure 3, we obtain

```
> error_approximation_gore(xt,zt,param
_equid(a,b,8), theta1, theta2,8,1);
Area of the surface development
approximation, 124044.6300
Surface of revolution  area,
125663.7062
Absolute Error, 1619.0762
Relative Error (%), 1.288419902
```

respectively,

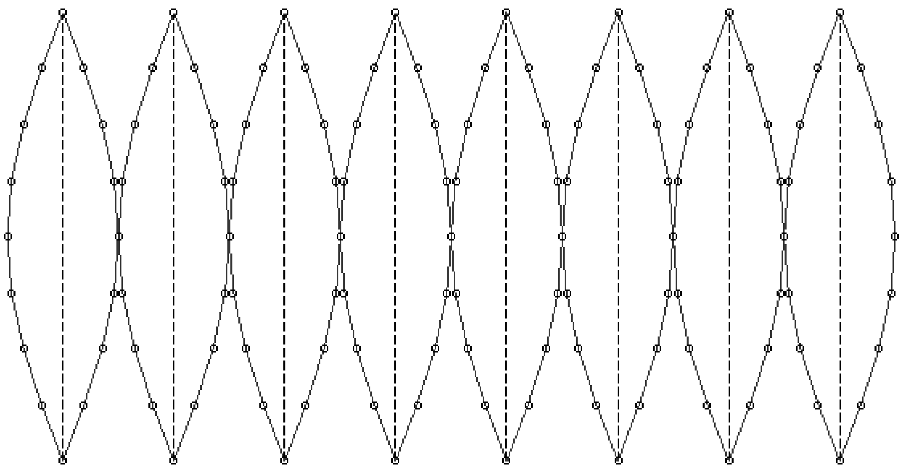

**Fig. 3** Sphere development, 8 gores (obtained using draw_development_gore [1])

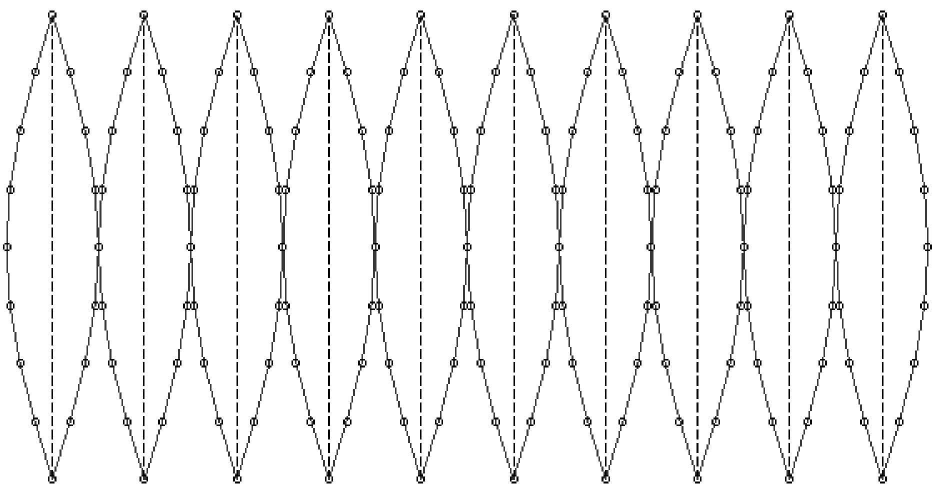

**Fig. 4** Sphere development, 10 gores (obtained using draw_development_gore [1])

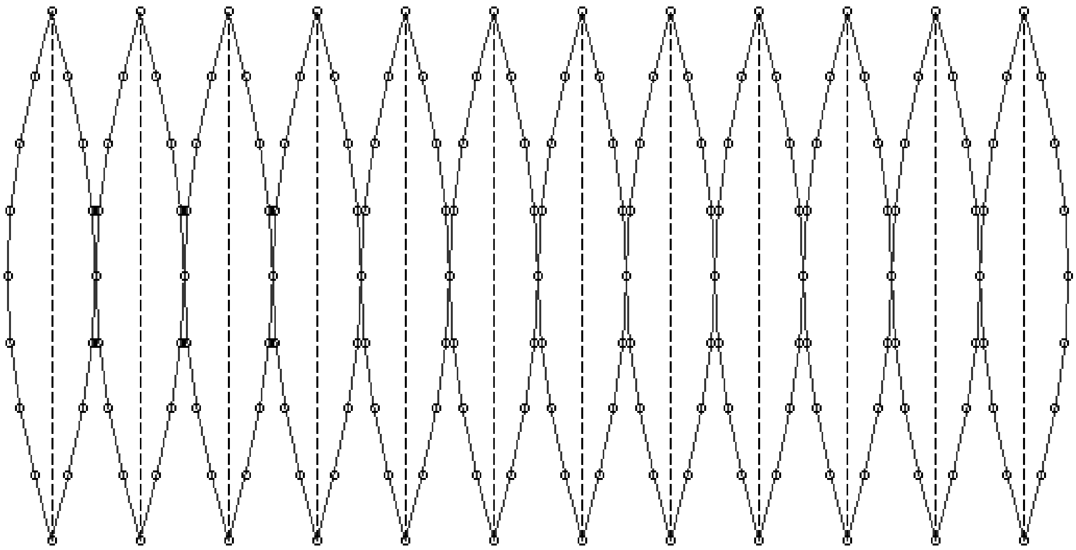

**Fig. 5** Sphere development, 12 gores (obtained using draw_development_gore [1])

```
>error_approximation_gore(xt,zt,param_
equid(a,b,8), theta1, theta2,8,3);
Area of the surface development
approximation, 125659.3230
Surface of revolution  area,
125663.7062
Absolute Error, 4.3832
Relative Error (%), 0.003488039731
```

In the case of an approximate development of sphere using gore method with 5 meridian (vertical) planes (resulting 10 gores), Figure 4, we obtain

```
>error_approximation_gore(xt,zt,param_
equid(a,b,8), theta1, theta2,10,1);
Area of the surface development
approximation, 124385.1289
Surface of revolution  area,
125663.7062
Absolute Error, 1278.5773
Relative Error (%), 1.017459487
```

respectively,

```
>error_approximation_gore(xt,zt,param_
equid(a,b,8), theta1, theta2,10,3);
Area of the surface development
approximation, 125661.0006
Surface of revolution  area,
125663.7062
Absolute Error, 2.7056
Relative Error (%), 0.002153048069
```

In the case of an approximate development of sphere using gore method with 6 meridian (vertical) planes (resulting 12 gores), Figure 5, we obtain

```
>error_approximation_gore(xt,zt,param_
equid(a,b,8), theta1, theta2,12,1);
Area of the surface development
approximation, 124385.1288
Surface of revolution  area,
125663.7062
Absolute Error, 1278.5774
Relative Error (%), 1.017459566
```

respectively,

```
>error_approximation_gore(xt,zt,
param_equid(a, b, 8), theta1,
theta2,12,3);
Relative Error (%), 0.002153127647
Area of the surface development
approximation, 125661.0005
Surface of revolution  area,
125663.7062
Absolute Error, 2.7057
Relative Error (%), 0.002153127647
```

Also for the same sphere, the approximate development is generated by zone method using successively: 7 level planes (8 zones), 9 level planes (10 zones) and 11 level planes (resulting in 12 zones). For every variant of development, the error is calculated by comparing the area of approximate development and the theoretical area of the sphere. Furthermore a computerized method based on Maple procedures is used. More precisely, to estimate the error, we use the procedure error_approximation_zones [2].

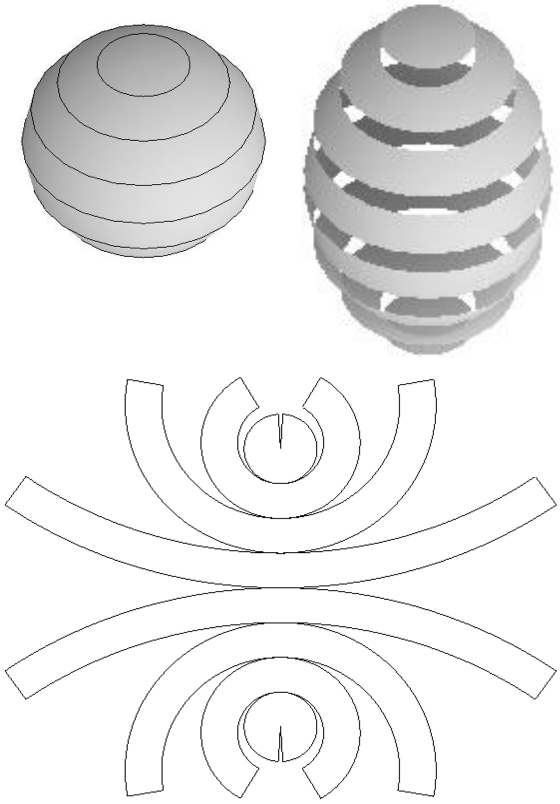

**Fig. 6** Sphere development, 8 zones (obtained using zones_approx, zones_approx_sepm and zones_draw [2])

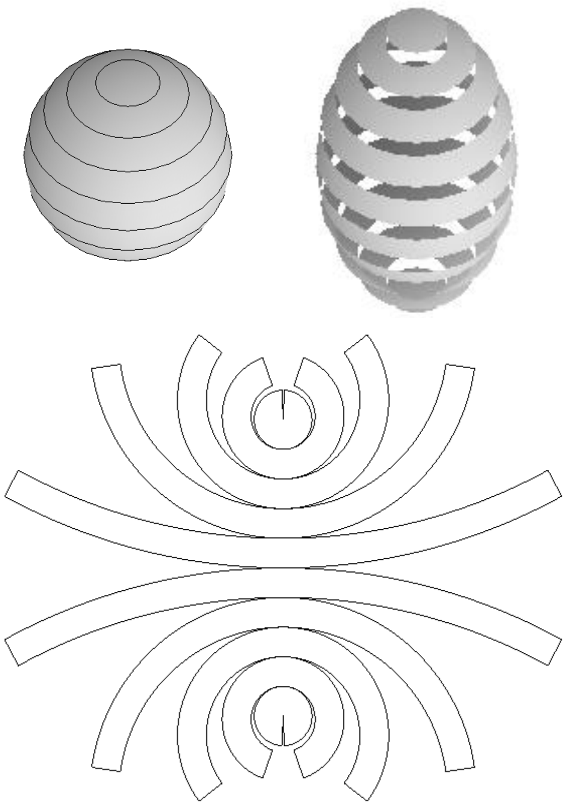

**Fig. 7** Sphere development, 10 zones (obtained using zones_approx, zones_approx_sepm and zones_draw [2])

In the case of the approximate development of sphere using zone method with 7 level planes (8 zones), Figure 6 (obtained using zones_approx, zones_approx_sepm and zones_draw [2]), the resulted error can be computed by:

```
>error_approximation_zones(xt,yt,
param_equid(a,b,8), theta1,theta2);

Area of the surface development
approximation, 123249.1133
Surface of revolution  area,
125663.7062
Absolute Error, 2414.59288
Relative Error (%), 1.921471962
```

In the case of the approximate development of sphere using zone method with 9 level planes (10 zones), Figure 7, we obtain

```
>error_approximation_zones(xt,yt,
param_equid(a,b,10), theta1,theta2);

Area of the surface development
approximation, 124116.5774
Surface of revolution area,
125663.7062
Absolute Error, 1547.128733
Relative Error (%), 1.231165927
```

In the case of the approximate development using zone method with 11 level planes (12 zones), Figure 8, it results:

```
>error_approximation_zones(xt,yt,
param_equid(a,b,12), theta1,theta2);

Area of the surface development
approximation, 124588.6358
Surface of revolution  area,
125663.7062
Absolute Error, 1075.070451
Relative Error (%), 0.8555138822
```

In the computations we used 10 significant digits (the number of digits carried in the mantissa for floating-point arithmetic). This means that the Maple environment variable Digits has the default value 10. Therefore the last decimal digit of the estimates is not relevant, being affected by rounding error (due to floating point arithmetic), as well as, by the fact that the integrals are computed by numerical methods.

## 4. CONCLUSION

Some pieces have geometric shapes formed by rotating surfaces that do not develop through exact methods. In these cases there occurs the problem of establishing the whole area by approximate methods. In many cases it is necessary that the whole area be established very precisely and then we must choose the most accurate calculation method, the smaller and smaller error method. Computerized methods offer the ability to draw the development of rotation surfaces with lower errors.

Let us summarize: in this paper we applied gore method, respectively zone method to obtain an approximate development of sphere. In order to estimate the error we compared the area of the original sphere and the area of the approximate development.

In the case of the gore method we used successively: 4 meridian (vertical) planes (resulting 8 gores), 5 meridian planes (resulting 10 gores) and 6 meridian planes (resulting 12 gores). The study highlighted the fact that the area of the approximate development does not depend on the number of gores, but on how well the gores “are approximated”. For joining the points obtained analogously to $A_{10}$, $B_{10}$, $C_{10}$, … (and symmetrically, $A_{20}$, $B_{20}$, $C_{20}$, …) in Figure 1, and thus to get a gore, we used spline functions of degree d, and we took into consideration two variants: d=1 and d=3. In case d=1, the gore is obtained by joining the points by lines. As we have seen in Section 3, the error is significantly smaller in the case d=3 than in the case d=1. More precisely, if we use 9 points to draw a gore, the relative error is approximately 1000 times lower in case d=3 than in case d=1. Moreover the errors strongly depend on the number of points used to approximate a gore. For instance in case d=1 if we use 11 points instead of 9, the relative error decreases from 1.288419902 % (see Section 3) to 0.8238232273 %:

```
>error_approximation_gore(xt,zt,param_
equid(a,b,10), theta1, theta2,8,1);

Area of the surface development
approximation, 124628.4594
Surface of revolution area,
125663.7062
Absolute Error, 1035.2468
Relative Error (%), 0.8238232273
```

In the case of the zone method we used successively: 7 level planes (resulting in 8 zones), 9 level planes (10 zones) and 11 level planes (12 zones). As we expected the approximate error decreases when the number of level planes increases.

As we have observed in Section 3, the error in the case of the zone method is significantly higher than that in the case of the gore method for d=3 (each gore is approximated by spline functions of degree 3). On the other hand, when 9 points are used to draw a gore, the resulted error in the case of the zone method for 8 zones (respectively, 10 zones or 12 zones) is higher (respectively, lower) than the error in the case of gore method for d = 1.  If instead of 9 points, 11 points are used to draw a gore, then the errors in case of zone method (for 8, 10 or 12 zones) are higher than those in the gore method.

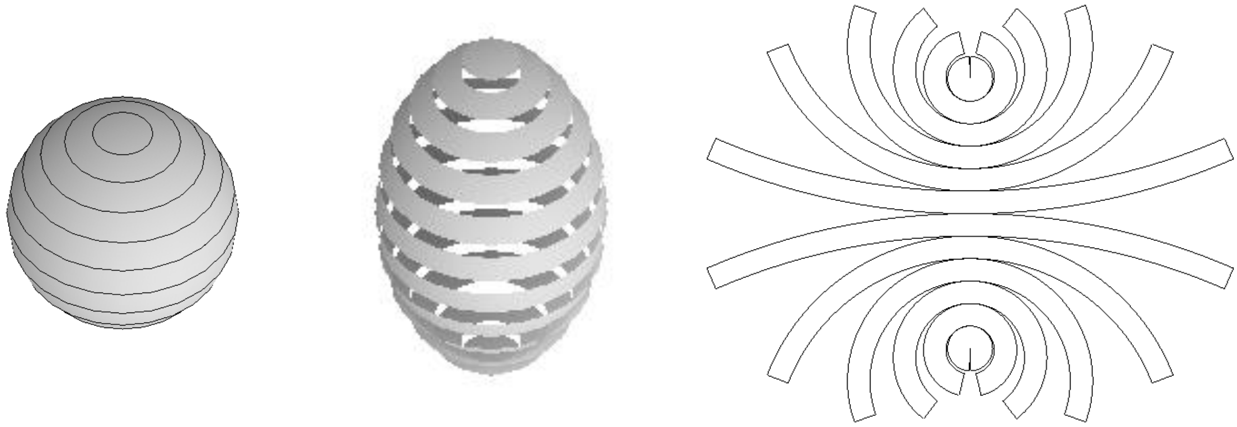

**Fig. 8** Sphere development, 12 zones (obtained using zones_approx, zones_approx_sepm and zones_draw [2])


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**Author:**

**Prof. Liliana LUCA**, Faculty of Engineering, University Constantin Brancusi of Targu-Jiu, Romania, E-mail: lylyanaluca@yahoo.com