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ASPECTS AND GEOMETRIC INTERPRETATION IN SOLVING CONICAL SECTIONS

Abstract: *In mathematics, a conic is the curve that is obtained by intersecting a plane with a cone. Is well known that the shape of this curve may differ quite a bit depending on the position of the intersection plane relative to the cone axis, as it is actually a family of curves, commonly called "conical". In the first part, we determine the relation between the shape of the section curve and the angle of the plane of the section plane. The paper also proposes a reassembly of the cone from the fragments resulting from the sectioning with different planes. These recommences can be found as technical solutions for joining two pipes with different diameters and whose axes are not coaxial.*

Key words: *conic, sections, decomposition, inclined surface, ellipse, parabola, hyperbola.*

1. INTRODUCTION

Considered one of the world's greatest geometer, Apollonius (262 – 190 BC) put forward various remarkable theories and fundamental papers (Figure 1, [1]). Special attention was paid to conic sections, including the parabola, the ellipse and the hyperbola. Apollonius work is still of inexhaustible importance for contemporary scholars and researchers, highly fruitful and innovative. Back in the ancient times, the Greek geometers focused mainly on laying out the figures selected from their inventory, which we abundantly apply nowadays in engineering and architecture projects.

As it stands as a plane curve, we can determine the conic section in various ways. In the Euclidean plane, we would consider it a set of points, connected with metric relations, whereas, in Cartesian plane, we would consider it a diagram of the special bilinear quadratic, determined by the analytic equation. Also, we would regard it as an image of a circle under a specific projective transformation.

Approaching a different perspective, Apollonius would cut the sections from oblique cones, hence displaying a more general case, while the right cone stands only a particular instance.

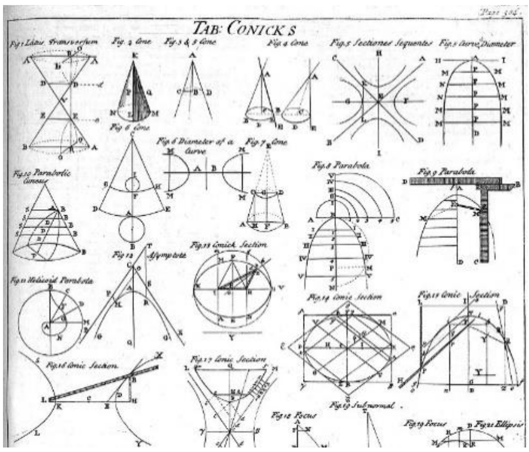

Fig.1. Page from Appolonius book [1].

2. THEORETICAL ASPECTS

We try to define a relation between the elements that define a cone, the angle α between the apparent generator and the radius of the same plane, the angle β between the plane that cuts the cone and the plane [xOy], which can lead to the general equation of a conic.[2]

We consider the coordinate system directly oriented, meaning the center of the cone base coincides with the origin of the coordinate system, and the cone tip is on the positive semiax Oz. (Figure 2)

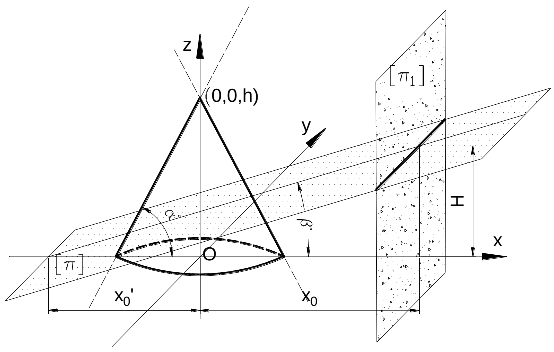

Fig.2. The right circular cone in the chosen coordinate system.

The coordinates of the cone tip are (0, 0, h). The equation of the circular conical surface is:

$$x^2 + y^2 = a(z-h)^2; a \in \Re; h \in \Re \tag{1}$$

Consider R - the radius of the circle determined by the intersection of the conical surface with the coordinate plane [xOy]. Then, as x and y are on the surface, $x^2 + y^2 = R^2$, for any $x, y \in [xOy]$,

So, for z=0 we obtain:

$$R^2 = a.h^2 \Rightarrow a = \frac{R^2}{h^2} \tag{2}$$

The equation of the conics becomes:

$$x^2+y^2=\frac{R^2}{h^2}(z-h)^2 \qquad (2)$$

If α $\in \Re$ is the magnitude of the angle formed by the cone base, then $tg\alpha = h/R$. We consider a plan [π_1] parallel to the plan [zOy], at a distance $x_0 \rangle 0$; the equation of this plan is: $x-x_0=0$. Consider another plane [π], parallel to the Oy axis, which intersects the plane [π1] after a line (parallel to the Oy axis), at a distance H from the plane [xOy].

We write the equation of plane [π] as a cluster of plane surfaces:

$$z=m(x-x_0)+H \quad , \qquad (3)$$

The planes that determines the cluster of plane surfaces are planes of equations x=x_0 and z=H.
We consider the angle $\beta = \angle([\pi],[xOy])$.

Then, if $H/(x_0-x_0^I)=tg\beta$ and x_0^I must verify the equation (3), we have: $0=m(x_0^I-x_0)+H$, meaning m=tgβ.
In this situation the equation of [π] plane is:

$$z=tg\beta(x-x_0)+H \qquad (4)$$

For the intersection of the plane [π] with the conic surface, we replace (4) in (2) and note that:

$$\frac{R^2}{h^2}=ctg^2\alpha \qquad (5)$$

We obtain:

$$(1-ctg^2\alpha.tg^2\beta)\,x^2+2.ctg^2\alpha.\,tg\beta(R\,tg\alpha-H+x_0\,tg\beta)x \\ -ctg^2\alpha\,(R.tg\alpha-H+x_0\,.tg\beta)^2=0 \qquad (6)$$

ie the equation of the intersection projected on the [xOy] plane in a general form.
Discussions:

- if $1-ctg^2\alpha\,.tg^2\beta \rangle 0$ - the intersection is elliptical; more precisely, if $1\rangle ctg^2\alpha\,.tg^2\beta$, so if $\alpha \rangle \beta$;
- if $1-ctg^2\alpha\,.tg^2\beta=0$ - the intersection is a parabola, $\alpha=\beta$;
- if $1-ctg^2\alpha\,.tg^2\beta \langle 0$ - the intersection is a hyperbola, so if $\alpha \langle \beta$.
- Remark 1. The equation of intersection, projected on [xOy] leads to the intersection from $\Re^3$ by considering the points $(x,y,z)\in\Re^3$, where (x, y) satisfy the equation (6), and z equation (4).
- Remark 2. The intersection projected on [xOy] is of the same type (ellipse, parabola, and hyperbola) with the intersection from $\Re^3$.
- Remark 3. If $1-ctg^2\alpha\,.tg^2\beta \neq 0$ then the conic of the intersection has a unique center, like the conic of the section projected [xOy].

We bring the conics in canonical form, obtaining:

$$(1-ctg^2\alpha.tg^2\beta)[x^2+2\frac{ctg^2\alpha.tg\beta}{1-ctg^2\alpha.tg^2\beta}(Rtg\alpha-H+x_0tg\beta)x \\ +(\frac{ctg^2\alpha.tg\beta}{1-ctg^2\alpha.tg^2\beta}(Rtg\alpha-H+x_0tg\beta))^2]+y^2- \\ (1-ctg^2\alpha.tg^2\beta)[\frac{ctg^2\alpha.tg\beta}{1-ctg^2\alpha.tg^2\beta}(Rtg\alpha-H+x_0tg\beta)]^2 \\ -ctg^2\alpha\,(R\ tg\alpha-H+x_0\ tg\beta)^2=\frac{ctg^2\alpha.}{ctg^2\alpha.tg^2\beta-1}\cdot \\ \cdot(R\,tg\alpha-H+x_0tg\beta)^2 \qquad (7)$$

The canonical form is:

$$(1-ctg^2\alpha.tg^2\beta)[x+\frac{ctg^2\alpha.tg\beta}{1-ctg^2\alpha.tg^2\beta}(Rtg\alpha-H+x_0tg\beta)]^2 \\ +y^2-\frac{ctg^2\alpha.}{1-ctg^2\alpha.tg^2\beta}(R\,tg\alpha-H+x_0tg\beta)^2=0 \qquad (8)$$

Or, equally, the canonical form of the conics projected on the plane [xOy] is:

$$[x+\frac{ctg^2\alpha.tg\beta}{1-ctg^2\alpha.tg^2\beta}(Rtg\alpha-H+x_0tg\beta)]^2+ \\ \frac{1}{1-ctg^2\alpha.tg^2\beta}y^2-\frac{ctg^2\alpha.}{1-ctg^2\alpha.tg^2\beta}\times \\ \times(R\,tg\alpha-H+x_0tg\beta)^2=0 \qquad (9)$$

The conics of intersection from $\Re^3$ are obtained from the system:

$$\begin{cases} (9) \\ z=tg\beta(x-x_0)+H \end{cases} \qquad (10)$$

If $1-ctg^2\alpha\,.tg^2\beta \neq 0$, then conics of the intersection from $\Re^3$ are unique centers, which are determined immediately from the canonical form, resulting:

$$x_C=-\frac{ctg^2\alpha.tg\beta}{1-ctg^2\alpha.tg^2\beta}(Rtg\alpha-H+x_0tg\beta) \\ y_C=0 \\ z_C=\frac{R\,tg\alpha.ctg^2\beta+x_0tg\beta-H}{ctg^2\alpha.tg^2\beta-1} \qquad (11)$$

and center C(x_C , y_C , z_C).

This allows us to "recompose" the cone after it has been cut, rotating it around the height to a new overlap.

Let

$$A = \frac{ctg^2\alpha.}{(1-ctg^2\alpha.tg^2\beta)^2}\,(Rtg\alpha - H + x_0 tg\beta)^2 \qquad (12)$$

And

$$\pm B^2 = \frac{1}{(1-ctg^2\alpha.tg^2\beta)\,A^2} \qquad (13)$$

Semi-axes A 'and B' of the conics with center, obtained by intersection compare to the semi axes A and B of the conics projection on the plane [xOy].

- on Oy: $B' = B$
- on Ox: $A' = A/\cos\alpha$

• In case of $1-ctg^2\alpha.tg^2\beta \langle 0$,

- the hyperbola projected on the [xOy] plane has asymptotes of equations $\frac{x^2}{A^2}-\frac{y^2}{B^2}=0$;
- the hyperbola obtained by the cone-plane intersection, so the hyperbola in $\Re^3$ has asymptotes of equations:

$$\begin{cases} \frac{x^2}{A^2}-\frac{y^2}{B^2}=0 \\ z = tg\beta(x-x_0)+H \end{cases} \qquad (14)$$

• In case of $1-ctg^2\alpha.tg^2\beta = 0$, so $tg\,\alpha = tg\beta$, we have:

$$\begin{aligned} &2.ctg^2\alpha.\ tg\beta(R\,tg\alpha - H + x_0\,tg\beta)x + y^2 - \\ &-ctg^2\alpha\,(R.tg\alpha - H + x_0\,.tg\beta)^2 = 0 \end{aligned} \qquad (15)$$

Or, equivalent

$$\begin{aligned} &y^2 + 2.ctg\alpha.(R\,tg\alpha - H + x_0\,tg\alpha)\cdot \\ &\cdot(x - \frac{R.tg\alpha - H + x_0\,.tg\alpha}{2\,tg\alpha}) = 0 \end{aligned} \qquad (16)$$

So, the tip of the conical projection on the [xOy] plane has the coordinates:

$$x_V = \frac{R.tg\alpha - H + x_0\,.tg\alpha}{2\,tg\alpha}$$

$$y_V = 0$$

$$z_V = tg\beta(x_V - x_0) + H = \frac{1}{2}\left(H + Rtg\alpha - x_0 tg\alpha\right) \qquad (17)$$

• Remark 4. If the surface is not a circular cone, but an elliptical cone, choosing in the same way the coordinate system and following the same construction, by analogue calculation, the equation

$$x^2 + y^2 = ctg^2\alpha.[tg\,\beta(x-x_0)+H-h]^2 \qquad (18)$$

It becomes:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = ctg^2\alpha.[tg\,\beta(x-x_0)+H-h]^2 \qquad (19)$$

and there are results analogous to those above, in which obviously, a and b will also appear the ellipse semi axes, representing the intersection of the conical surface with the plane [xOy].

• Remark 5. Various values can be given to the variables that occur, namely: $\alpha., \beta, h, H, R, x_0$, depending on the purpose proposed, so that quantitative and qualitative assessments can be made for their increases. If the surface is an elliptical, variable values can be assigned to *a* and *b*.

3. RECOMBINATIONS OF THE SECTIONED CONE

Depending on the angles α and β defined above, a relationship can be found between these and the angle γ, the angle of the initial height of the cone with the new elliptical base (Figure 3), then ε - the angle obtained by recomposing the cone after the plane of the section and rotation around its own axis.

We obtained the relations:

$$\gamma = 90^0 + \beta \quad \text{and} \quad \varepsilon = 90^0 - 2\beta \qquad (20)$$

So, from all parameters, the only parameter that influences the desired result is the angle β – the angle below which section is made.

Depending on what we want to achieve by recomposing the cone, we can determine the angle at which the cone will be sectioned or vice versa.

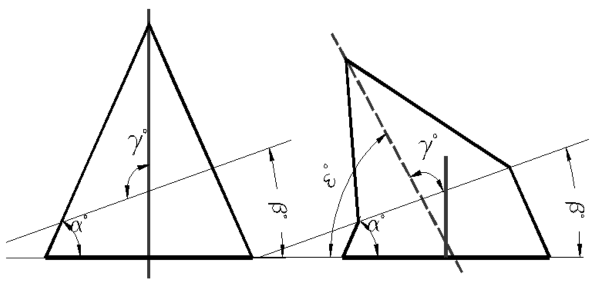

Fig.3. The angles in the sectioned cone with a plane.

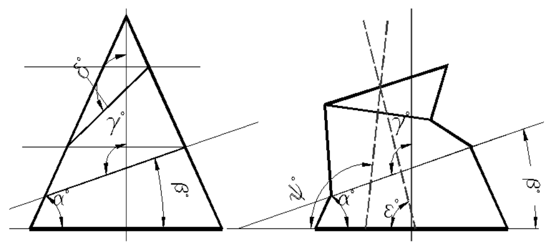

Fig.4. The angles in the sectioned cone with two planes.

With the cone's 3D modeling and cutting it with various inclined planes, we can obtain some truncated conical surfaces delimited by the ellipses section, as in Figure 5 and Figure 6.[3]

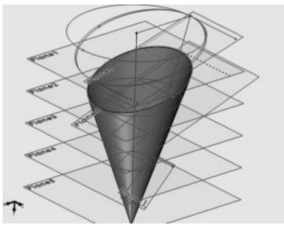

Fig. 5 Cone segmented with different planes.

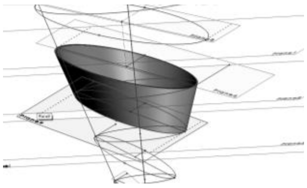

Fig. 6 Sector of truncated cone.

After sectioning, the cone can be reconstructed from these truncated segments of the cone, attaching them to the resulting elliptical section. Various variants are shown in Figure 7, Figure 9 and Figure 10, which can be used in practice in situations where we need a change of direction and a reduction in the connection diameter of different types of pipes.[4]

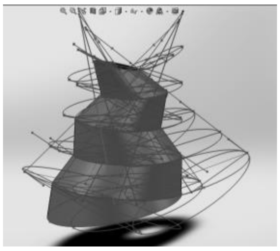

Fig. 7 Alternative 1 of recombination.

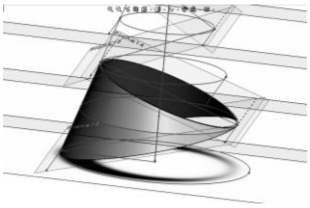

Fig. 8 Cone truncated sector.

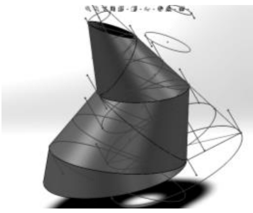

Fig. 9 Alternative 2 of recombination

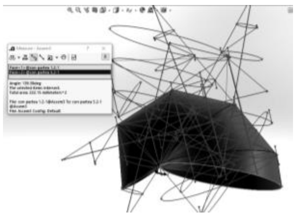

Fig. 10 Alternative 3 of recombination.

4. CONCLUSION

Conical sections are theoretically regarded as solutions of a system of equations under imposed conditions. The obtained equations are based on the cone parameters and the position of the section plane, relative to a particular coordinate system.

The work attempts to find various variants of cone recomposition, leaving imagination to create as many possibilities of joining as possible, taking different forms. Various forms can be generated by combining sectors. Some initials can be preserved, others can be rotated around the axes, the contact surface being the result of sloping with the inclined plane.

We can find a relations between the angle under which the cone is cut and the angle under which the direction shift can be made by recomposing it, depending on what we want to achieve.

The angles determined for each recomposition case are centralized in the tables below.

Table 1

Angles measured for alternative 1.

The angle between cutting plane and the horizontal plane	**The angle between the axis of each truncated cone (fig.7)**
18.34^0	30.51^0
20.23^0	6.37^0
30.96^0	40.03^0
45^0	21.90^0

Table 2

Angles measured for alternatives 2 and 3.

The angle between cutting plane and the horizontal plane	**The angle between the axis of each truncated cone (fig.9)**	**The angle between the axis of each truncated cone (fig.10)**
33.69^0	30.51^0	30.51^0
41.63^0	30.51^0	67.38^0
54.16^0	15.89^0	66.22^0
75.96^0	15.89^0	4.30^0

If the sections are made at very small distances between planes, then the cone surface of the tapered trunks will be nearly curved.

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