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GEOMETRIC STUDY ON TRANSFORMATIONS THROUGH SYMETRY IN 3D SPACE

Abstract: *In this paper, we present an analysis of the symmetry of the point and of the line in relation to the most known curved surfaces: cylindrical, conical and spherical.*
This analysis offers mathematical formulas showing the result of the transformations through symmetry and displays these changes through images obtained by spatial symmetrization, realized with Inventor software.
These results can be used in graphical applications assisted by computers, which use the symmetry in relation to curved surfaces.

Key words: *curved surfaces, symmetry, mathematical model, cylindrical surface, sphere, conical surface*

1. INTRODUCTION

In this paper we plan to study the symmetrical of a point/line in relation to the most known curved surfaces: conical, cylindrical, spherical. Their study is justified by the fact that any curved complex surface can be represented by a surface of this kind. These surfaces present the advantage that they offer the symmetrical of any point of space. The difficulty of symmetry in relation to a surface is due to the fact that there are surfaces that don't allow finding a symmetry of a point.

The symmetry of a point in relation to a point/line/plan is well known in classical geometry.

In the case of symmetry in relation with a curved surface, the problem is more complicated. In this paper, we realize a mathematical and graphical study in order to come with a solution for this problem.

So, we consider the projection of a point P on a surface, as being any point Q of the surface, in which the perpendicular on this surface passes through the point P. If P' is the symmetrical of P in relation with Q, then, we may say that P' is the symmetrical of P in relation to the surface.

It is possible that on the surface it may not exist a perpendicular able to pass through the point P. In this case, the point P doesn't have a symmetrical in relation to the surface.

It is also possible that it may exist many points of the surface, situated on the perpendicular on the surface that passes through the point P. In this case, we can say that the point P has many symmetrical points in relation with the surface. For example, in relation with the surfaces we have chosen in this paper, any point from space has two symmetrical points.

2. THE SYMMETRY OF A LINE IN RELATION TO A CYLLINDRICAL SURFACE

We consider the equation of a line (d) (noted by $\overline{D}$ in Fig.1)

$$\begin{aligned} x &= m_1 t + x_0 \\ y &= m_2 t + y_0 \\ z &= m_3 t + z_0 \end{aligned} \quad , \; t \in R \qquad (1)$$

where (x_0, y_0, z_0) - a point through which (d) passes, (m_1, m_2, m_3) - the directions of the line (d).

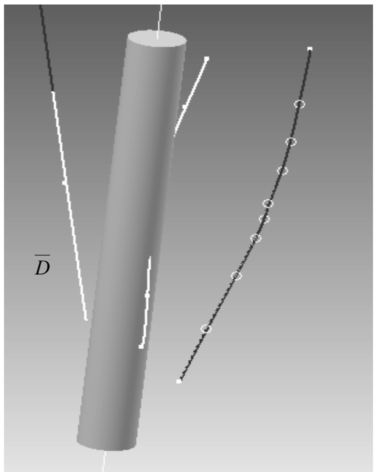

Fig. 1 The symmetry of a line in relation to a cylindrical surface.

The cylinder has the equation

$$\begin{cases} x = r\cos\alpha \\ y = r\sin\alpha \\ z \in R \end{cases} , \; \alpha \in [0, 2\pi] \qquad (2)$$

We consider a point $P(x_1, y_1, z_1) \in (d)$.

The perpendicular on the cylinder surface, which passes through the point, P is the line which passes through the point $O(0,0,z_1)$ which belongs to the cylinder axis and through P.

The intersection of this line with the cylinder is given by the solution of the system:

$$\begin{cases} r\cos\alpha = x_1(1-\lambda) \\ r\sin\alpha = y_1(1-\lambda) \\ z = z_1 \end{cases} \qquad (3)$$

$$r^2\cos^2\alpha + r^2\sin^2\alpha = x_1^2(1-\lambda)^2 + y_1^2(1-\lambda)^2$$

$$r^2\left(\cos^2\alpha + \sin^2\alpha\right) = \left(x_1^2 + y_1^2\right)\cdot(1-\lambda)^2 \quad \Rightarrow$$

$$1-\lambda=\pm\frac{r}{\sqrt{x_1^2+y_1^2}} \quad \Rightarrow \quad \lambda=1\mp\frac{r}{\sqrt{x_1^2+y_1^2}}$$

Let A and B be the points in which the perpendicular to the surface of the cylinder intersects the cylinder. The point A has the following coordinates :

$$\begin{cases} x_A=x_1\dfrac{r}{\sqrt{x_1^2+y_1^2}} \\ y_A=y_1\dfrac{r}{\sqrt{x_1^2+y_1^2}} \\ z_A=z_1 \end{cases} \tag{4}$$

The point B has the following :

$$\begin{cases} x_B=x_1\dfrac{-r}{\sqrt{x_1^2+y_1^2}} \\ y_B=y_1\dfrac{-r}{\sqrt{x_1^2+y_1^2}} \\ z_B=z_1 \end{cases} \tag{5}$$

Let S_B be the symmetry of P in relation to the point B:

$$\begin{cases} x_{S_B}=2x_1\left(-\dfrac{r}{\sqrt{x_1^2+y_1^2}}\right)-x_1 \\ y_{S_B}=2y_1\left(-\dfrac{r}{\sqrt{x_1^2+y_1^2}}\right)-y_1 \\ z_{S_B}=2z_1-z_1 \end{cases} \tag{6}$$

Considering $P(x_1,y_1,z_1)$ as a variable on the line (d) and replacing x_1,y_1,z_1 by x,y,z from the parametrical equation of the line (d) in the system from above, it results that :

$$\begin{cases} x=(m_1t+x_0)\cdot\left(-1-\dfrac{2r}{\sqrt{(m_1t+x_0)^2+(m_2t+y_0)^2}}\right) \\ y=(m_2t+y_0)\cdot\left(-1-\dfrac{2r}{\sqrt{(m_1t+x_0)^2+(m_2t+y_0)^2}}\right) \\ z=m_3t+z_0 \end{cases} \tag{7}$$

where $t\in R$, is the parametrical equation of the first curve symmetrical with (d) in relation to the cylindrical surface.

Similarly, we obtain the parametrical representation of the second curve, which is symmetrical with (d) in relation to the cylindrical surface, with the equations:

$$\begin{cases} x=(m_1t+x_0)\cdot\left(-1+\dfrac{2r}{\sqrt{(m_1t+x_0)^2+(m_2t+y_0)^2}}\right) \\ y=(m_2t+y_0)\cdot\left(-1+\dfrac{2r}{\sqrt{(m_1t+x_0)^2+(m_2t+y_0)^2}}\right) \\ z=m_3t+z_0 \end{cases} \tag{8}$$

3. SYMMETRY OF A LINE IN RELATION TO A SPHERE

We consider a sphere having the equation:

$$x^2+y^2+z^2=R^2 \tag{9}$$

The equation of the line (d) (noted by $\overline{D}$ in Fig. 2):

$$\begin{cases} x=m_1t+x_0 \\ y=m_2t+y_0 \\ z=m_3t+z_0 \end{cases},\quad t\in R \tag{10}$$

where (x_0,y_0,z_0) = a point through which (d) passes and (m_1,m_2,m_3) = the parameters of the line (d).

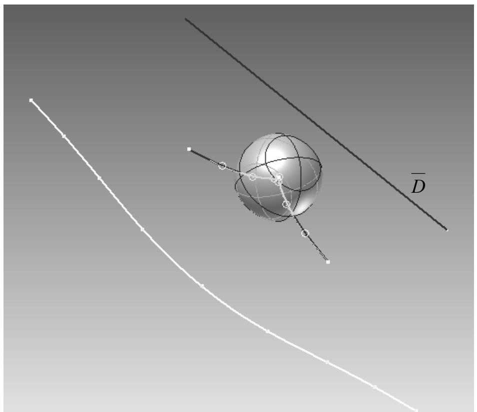

Fig. 2 Symmetry of a line in relation to a sphere.

The perpendicular (d') on the sphere's surface, which passes through $P(x_1,y_1,z_1)\in(d)$ and through $O(0,0,0)$ will have the equation

$$\begin{cases} x=x_1+\lambda(0-x_1) \\ y=y_1+\lambda(0-y_1) \\ z=z_1+\lambda(0-z_1) \end{cases},\quad \lambda\in R$$

$$\Rightarrow \begin{cases} x=x_1(1-\lambda) \\ y=y_1(1-\lambda) \\ z=z_1(1-\lambda) \end{cases} \tag{11}$$

The intersection of (d') with the sphere will coincide with the solution of the following system

$$\begin{cases} x^2+y^2+z^2=R^2 \\ x=x_1-\lambda x_1 \\ y=y_1-\lambda y_1 \\ z=z_1-\lambda z_1 \end{cases} \tag{12}$$

Let A and B be the points at which (d') intersects the sphere.

The point B has the coordinates :

$$\begin{cases} x_B = x_1\left(-\dfrac{R}{\sqrt{x_1^2+y_1^2+z_1^2}}\right) \\ y_B = y_1\left(-\dfrac{R}{\sqrt{x_1^2+y_1^2+z_1^2}}\right) \\ z_B = z_1\left(-\dfrac{R}{\sqrt{x_1^2+y_1^2+z_1^2}}\right) \end{cases} \quad (13)$$

The point S_B - the symmetrical of P in relation to B will have the coordinates:

$$\begin{cases} x_{S_B} = x_1\left(\dfrac{-2R}{\sqrt{x_1^2+y_1^2+z_1^2}}-1\right) \\ y_{S_B} = y_1\left(\dfrac{-2R}{\sqrt{x_1^2+y_1^2+z_1^2}}-1\right) \\ z_{S_B} = z_1\left(\dfrac{-2R}{\sqrt{x_1^2+y_1^2+z_1^2}}-1\right) \end{cases} \quad (14)$$

Because $P(x_1, y_1, z_1)$ is some point from the line (d) the first symmetrical curve with (d) in relation to the sphere, will have the parametrical representation by replacing x_1, y_1, z_1 by x, y, z from the equation (10).

So, we obtain:

$$\begin{cases} x = (m_1t+x_0)\cdot\left(\dfrac{-2R}{\sqrt{(m_1t+x_0)^2+(m_2t+y_0)^2+(m_3t+z_0)^2}}-1\right) \\ y = (m_2t+y_0)\cdot\left(\dfrac{-2R}{\sqrt{(m_1t+x_0)^2+(m_2t+y_0)^2+(m_3t+z_0)^2}}-1\right) \\ z = (m_3t+z_0)\cdot\left(\dfrac{-2R}{\sqrt{(m_1t+x_0)^2+(m_2t+y_0)^2+(m_3t+z_0)^2}}-1\right) \end{cases} \quad (15)$$

Analogically, we calculate the coordinates of S_A - the symmetrical of P in relation to A, which will have the coordinates:

$$\begin{cases} x_{S_A} = x_1\left(\dfrac{2R}{\sqrt{x_1^2+y_1^2+z_1^2}}-1\right) \\ y_{S_A} = y_1\left(\dfrac{2R}{\sqrt{x_1^2+y_1^2+z_1^2}}-1\right) \\ z_{S_A} = z_1\left(\dfrac{2R}{\sqrt{x_1^2+y_1^2+z_1^2}}-1\right) \end{cases} \quad (16)$$

Therefore, the parametrical equation of the second symmetrical curve of the line (d) in relation to the sphere will have the parametrical representation:

$$\begin{cases} x = (m_1t+x_0)\left(\dfrac{2R}{\sqrt{(m_1t+x_0)^2+(m_2t+y_0)^2+(m_3t+z_0)^2}}-1\right) \\ y = (m_2t+y_0)\left(\dfrac{2R}{\sqrt{(m_1t+x_0)^2+(m_2t+y_0)^2+(m_3t+z_0)^2}}-1\right) \\ z = (m_3t+z_0)\left(\dfrac{2R}{\sqrt{(m_1t+x_0)^2+(m_2t+y_0)^2+(m_3t+z_0)^2}}-1\right) \end{cases} \quad (17)$$

4. SYMMETRY OF A LINE IN RELATION TO A CONICAL SURFACE

We consider $x^2+y^2-\left(R\dfrac{z}{z_0}\right)^2=0$ the equation of the conical surface, which passes through $O(0,0,0)$ and lean on the circle

$$\begin{cases} x^2+y^2=R^2 \\ z=z_0 \end{cases} \quad (18)$$

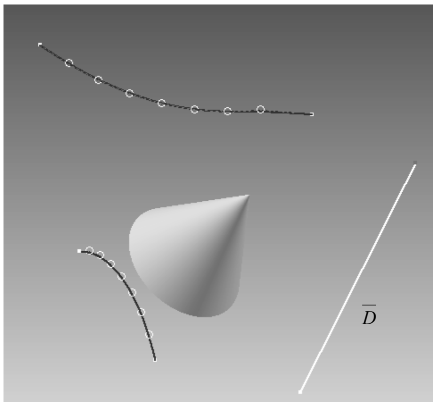

Fig. 3 The symmetry of the line in relation to a conical surface.

We note $F(x,y,z)=x^2+y^2-\left(R\dfrac{z}{z_0}\right)^2$ (19)

The perpendicular on the conical surface, in a point called (x_C, y_C, z_C) that has the equation:

$$\frac{x-x_C}{F'_x(x_C,y_C,z_C)}=\frac{y-y_C}{F'_y(x_C,y_C,z_C)}=\frac{z-z_C}{F'_z(x_C,y_C,z_C)}$$

or

$$\begin{cases} x = x_C+F'_x(x_C,y_C,z_C)\cdot t \\ y = y_C+F'_y(x_C,y_C,z_C)\cdot t \\ z = z_C+F'_z(x_C,y_C,z_C)\cdot t \end{cases} \quad (20)$$

Putting the condition that this perpendicular to the conical surface to pass through $P(x_1, y_1, z_1)$, situated on the line (d) (noted by $\overline{D}$ in Fig. 3) it results that:

$$\begin{cases} x_1 = x_C+F'_x(x_C,y_C,z_C)\cdot t \\ y_1 = y_C+F'_y(x_C,y_C,z_C)\cdot t \\ z_1 = z_C+F'_z(x_C,y_C,z_C)\cdot t \end{cases} \quad (21)$$

We add to this relation, the one that verifies (x_C, y_C, z_C) as a point of the conical surface and we obtain the following system

$$\begin{cases} x_1 = x_C + F'_x(x_C, y_C, z_C)\cdot t \\ y_1 = y_C + F'_y(x_C, y_C, z_C)\cdot t \\ z_1 = z_C + F'_z(x_C, y_C, z_C)\cdot t \\ x_C^2 + y_C^2 - R^2 \dfrac{z_C^2}{z_0^2} = 0 \end{cases} \tag{22}$$

By solving this system, we find two points – noted by C_1 and C_2 belonging to the conical surface, at which the perpendicular to the surface passes through the point $P(x_1, y_1, z_1)$, points which have the coordinates:

$$\begin{cases} x_{C_{1,2}} = \dfrac{x_1}{1+2t_{1,2}} \\ y_{C_{1,2}} = \dfrac{y_1}{1+2t_{1,2}} \\ z_{C_{1,2}} = \dfrac{z_1}{1-2\dfrac{R^2}{z_0^2}t_{1,2}} \end{cases} \tag{23}$$

where

$$t_{1,2} = \frac{R^2\left(x_1^2+y_1^2+z_1^2\right)}{2\left[\dfrac{R^4}{z^2}\left(x_1^2+y_1^2\right)-R^2z_1^2\right]} \pm$$

$$\pm \frac{\sqrt{R^2\left(x_1^2+y_1^2\right)\left[z_1^2\left(2R^2+z_0^2\right)+R^2\dfrac{z_1^2}{z_0^2}\right]}}{2\left[\dfrac{R^4}{z^2}\left(x_1^2+y_1^2\right)-R^2z_1^2\right]} \tag{24}$$

Considering that S_{C_1} as the symmetrical point of $P(x_1, y_1, z_1)$ in relation to the point $C_1(x_{C_1}, y_{C_1}, z_{C_1})$ and S_{C_2} the symmetrical of P in relation to $C_2(x_{C_2}, y_{C_2}, z_{C_2})$, they will have the coordinates

$$\begin{cases} x_{S_{C_{1,2}}} = x_1\dfrac{1-2t_{1,2}}{1+2t_{1,2}} \\ y_{S_{C_{1,2}}} = y_1\dfrac{1-2t_{1,2}}{1+2t_{1,2}} \\ z_{S_{C_{1,2}}} = z_1\dfrac{1+2\dfrac{R^2}{z_0^2}\cdot t_{1,2}}{1-2\dfrac{R^2}{z_0^2}\cdot t_{1,2}} \end{cases} \tag{25}$$

We consider that the line (d) on which the point $P(x_1, y_1, z_1)$ is situated, is a line given by the parametrical equations:

$$\begin{cases} x = x_2 + m_1\lambda \\ y = y_2 + m_2\lambda \\ z = z_3 + m_3\lambda \end{cases} \tag{26}$$

where (x_2, y_2, z_2) is a point belonging to (d), (m_1, m_2, m_3) are the parameters of the line and $\lambda \in R$.

Considering the floating point P on (d) then S_{C_1} and S_{C_2} will describe two curves that will represent the symmetricals of (d) in relation to the conical surface.
By replacing x_1, y_1, z_1 in the relations (24) by x, y, z from the relations (26) we obtain t_1 and t_2 depending on λ.

Also, in relations (25) we replace x_1, y_1, z_1 by x, y, z from the relations (26), and t_1 and t_2 by t_1 and t_2 obtained based on λ and so, we obtain the parametrical representation for the curves symmetricals with (d) in relation to the cylindrical surface, in the form of :

$$\begin{cases} x = (x_2 + m_1\lambda)\cdot\dfrac{1-A}{1+A} \\ y = (y_2 + m_2\lambda)\cdot\dfrac{1-A}{1+A} \\ z = (z_2 + m_3\lambda)\cdot\dfrac{z_0^2 + R^2\cdot A}{z_0^2 - R^2\cdot A} \end{cases} \tag{27}$$

where:

$$A = \Big\{ R^2\left[(x_2+m_1\lambda)^2+(y_2+m_2\lambda)^2+(z_2+m_3\lambda)^2\right] \pm$$
$$\sqrt{R^2\left[(x_2+m_1\lambda)^2+(y_2+m_2\lambda)^2\right]\cdot\left[(z_2+m_3\lambda)^2\left(2R^2+z_0\right)^2+R^4\cdot(z_2+m_3\lambda)^2\right]:z_0^2}\ \Big\}:$$
$$\left\{R^4\left[(x_2+m_1\lambda)^2+(y_2+m_2\lambda)^2-R^2(z_2+m_2\lambda)^2-R^2(z_2+m_3\lambda)^2\right]:z_0^2\right\}$$

and $\lambda \in R$.

5. CONCLUSION

The results from this study are presented in an algorithmic form, easy to use, as we have previously specified, in graphical applications on the computer.

The results may present a practical interest in the study of images, or only a geometrical interest.

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